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A State-Dependent Riccati Equation Approach to Atmospheric Entry Guidance

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This paper investigates the use of state-dependent Riccati equation control for closed-loop guidance of the hypersonic phase of atmospheric entry. Included are a discussion of the development of the state-dependent Riccati equations, their outgrowth from Hamilton-Jacobi-Bellman theory, a discussion of the closed-loop nonlinear system's closed-loop stability and robustness from both a theoretical and practical viewpoint. An innovative use of sum-of-squares programming is used to solve the state-dependent Riccati equation with application of a state-dependent Riccati equation derived guidance algorithm to a high mass, robotic Mars entry example. Algorithm performance is compared to the Modified Apollo Final Phase algorithm planned for use on the Mars Science Laboratory.

I. Motivation

When deciding where spacecraft should land a balance exists between engineering safety and scientific (or programmatic) interest. As safe, interesting landing sites become more sparse and accuracy requirements become more stringent, the need for guidance algorithms that are capable of targeting full state (*i.e.*, fully specified position and velocity vectors, $\mathbf{r}_{3\times1}$ and $\mathbf{v}_{3\times1}$) for the (1) hypersonic and (2) terminal descent phase becomes more pervasive.[1,2] For the hypersonic aeromaneuvering phase, these target conditions are typically the state at the deployment of the supersonic decelerator; whereas, for the propulsive terminal descent case, the end conditions are the final touchdown state of the vehicle. Of course, when implementing a control law, it is desirable to minimize a certain quantity, which most commonly in entry guidance problems is the control effort (*i.e.*, the propellant mass). In the Apollo program, as with the planned Mars Science Laboratory, hypersonic guidance is performed using a tracking controller which commands a given bank angle based on linear perturbations from a predefined reference path to maneuver the vehicle towards the reference path.[3,4] The reference path is designed and tuned to achieve desired drag, deceleration, and heating profiles while achieving the target performance without directly optimizing the fuel consumption of the reaction control system required to command these bank angles. Similarly, the future Crew Exploration Vehicle currently does not include propellant usage directly in a cost function as it is baselined to utilize a predictor-corrector approach to hypersonic aeromaneuvering to target the initial state of the reference path guidance scheme of Apollo.[5] Given the highly nonlinear dynamics of atmospheric entry, developing a guidance algorithm that optimizes a performance index is difficult, particularly one that has implementation potential onboard a flight system.

II. Overview

The use of the state-dependent Riccati equation (SDRE) is an emerging way to provide feedback control for nonlinear systems that was initial proposed by Pearson in 1962.[6] While initially proposed in 1962, the real emergence of interest in the field did not come until the late 1990's when several studies by Cloutier, D'Souza, and Mracek showed the applicability and promise of the SDRE to nonlinear control of aerospace systems.[7–9] The SDRE control method employs factorization of the nonlinear dynamics into a state vector and state dependent matrix valued function that is capable of capturing the nonlinear system dynamics by a linear system that is dependent on the current state.[9] Through Hamilton-Jacobi-Bellman theory, the minimization of a linear system with a quadratic performance index is possible resulting in an algebraic Riccati equation (ARE) in terms of the state dependent matrices that are the resultant of the factorization of the system. Therefore, a slightly sub-optimal feedback control law is able to be obtained

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through solution of the ARE at each point in state space. The SDRE for infinite-time problems with affine control in the dynamics and quadratic control in the cost has been shown to asymptotically stable and locally asymptotically optimal; however, asymptotically the Pontryagin optimal necessary conditions are satisfied for the general case.[7,8] It should also be mentioned that because the factorization of the nonlinear system is non-unique, the flexibility afforded to the control system designer allows for additional control system robustness to be incorporated. As such, various decomposition techniques can be employed, exploiting the factorization yielding the most robust closed-loop system. The analysis of the stability and robustness of the open- and closed-loop system can be conducted using a sum-of-squares approach where polynomials are used to approximate the Lyapunov function.[10] The sum-of-squares technique generalizes linear matrix inequalities to form a semi-definite programming problem which can be used to impose stability requirements on the system. These requirements can further be extended to determine the robustness of the system to perturbations by including these perturbations and parameters of the system.

III. State-Dependent Riccati Regulation Theory

A. The General Nonlinear Regulation Problem

To begin, consider the autonomous system that is full-state observable and affine in input (control) given by

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x_0} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$ is the state vector, $\mathbf{u} \in \mathbb{R}^{\mathbf{m}}$ is the input vector, and $t \in [0, \infty)$ is the time. Additionally, it is assumed that $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $\mathbf{B} : \mathbb{R}^n \mapsto \mathbb{R}^{n \times m}$ with $\mathbf{B}(\mathbf{x}) \neq \mathbf{0}\ \forall \mathbf{x}$ are in $\mathcal{C}^1$ and that the origin of the uncontrolled, nonlinear system is an equilibrium. Formulated in terms of an optimal regulation problem, it is desired to minimize the infinite-time performance index

$$J\left(\mathbf{x}(t), \mathbf{u}(t)\right) = \frac{1}{2}\int_0^\infty (\mathbf{x}^T(t)\mathbf{Q}(\mathbf{x})\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}(\mathbf{x})\mathbf{u}(t))dt \tag{2}$$

which is not necessarily quadratic in state, but is quadratic in input. Furthermore, the weighting matrices satisfy $\mathbf{Q}(\mathbf{x}) \geq 0$ and $\mathbf{R}(\mathbf{x}) > 0$ for all $\mathbf{x}$. The optimal regulation problem is then to find a control law of the form,

$$\mathbf{u}(\mathbf{x}) = -\mathbf{K}(\mathbf{x})\mathbf{x}, \quad \mathbf{u}(\mathbf{0}) = \mathbf{0} \tag{3}$$

such that the performance index is minimized subject to systems dynamics (1) while the closed-loop system is driven to the origin in infinite time, that is

$$\lim_{t \to \infty} \mathbf{x}(t) = \mathbf{0} \tag{4}$$

B. Extended Linearization of the Nonlinear Dynamics

Consider the state-dependent coefficient (SDC) matrix factorization of the input-free nonlinear system, in which the nonlinear system is factorized into a linear like structure containing matrices which depend on the state. That is, factorizing the system such that the nonlinear system, (1), becomes[7]

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x})\mathbf{x}(t) + \mathbf{B}(\mathbf{x})\mathbf{u}(t) \quad \mathbf{x}(0) = \mathbf{x_0} \tag{5}$$

where $\mathbf{A} : \mathbb{R}^n \mapsto \mathbb{R}^{n \times n}$ is only unique in the case where $n = 1$. This factorization of $\mathbf{f}(\mathbf{x})$ is guaranteed to exist provided that $\mathbf{f}(0) = \mathbf{0}$ and $\mathbf{f}(\mathbf{x}) \in \mathcal{C}^1$ The SDC factorized system now appears to be linear system; however, $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ depend on the state, making the system nonlinear.

C. State-Dependent Riccati Equation Control

Using the linear-like structure of (5) as a solution direction, the regulation of the nonlinear system can be formulated similarly to a linear-quadratic regulator (LQR) problem with cost function given by (2). This results in a state-dependent ARE, or SDRE, of the form[9]

$$\mathbf{P}(\mathbf{x})\mathbf{A}(\mathbf{x}) + \mathbf{A}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) - \mathbf{P}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) + \mathbf{Q}(\mathbf{x}) = \mathbf{0} \tag{6}$$

where $\mathbf{P}(\mathbf{x}) > 0$ is the unique, symmetric matrix being sought. Provided $\mathbf{P}(\mathbf{x})$ exists, the feedback control law is then

$$\mathbf{u}(\mathbf{x}) = -\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^{\mathbf{T}}(\mathbf{x})\mathbf{P}(\mathbf{x})\mathbf{x} \tag{7}$$

where the feedback gain matrix, $\mathbf{K}(\mathbf{x})$ is given by

$$\mathbf{K}(\mathbf{x}) = -\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^{\mathbf{T}}(\mathbf{x})\mathbf{P}(\mathbf{x}) \tag{8}$$

Enacting this control law, results in the closed-loop system dynamics being given by

$$\dot{\mathbf{x}}(t) = \left[\mathbf{A}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^{\mathbf{T}}(\mathbf{x})\mathbf{P}(\mathbf{x})\right]\mathbf{x}(t) \tag{9}$$

Hence, the SDRE control solution to the nonlinear problem is just a generalization of the the infinite-time LQR problem, where instead of constant matrices, the matrices are now state-dependent.

D. The Connection to Hamilton-Jacobi-Bellman Theory

Hamilton-Jacobi-Bellman (HJB) theory provides the mathematical conditions in order to the optimum feedback control law under the dynamic programming framework. With $\mathbf{f}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$, $\mathbf{Q}(\mathbf{x})$, and $\mathbf{R}(\mathbf{x})$ sufficiently smooth, the value function, defined as[15]

$$V(\mathbf{x}) \triangleq \inf_{\mathbf{u}(\cdot)\in U} J\left(\mathbf{x}, \mathbf{u}(\cdot)\right) \tag{10}$$

is continuously differentiable over the set of admissible controls, U. A solution is then sought for the value function that is stationary and satisfies the partial differential equation

$$\frac{\partial}{\partial t}V(\mathbf{x}) + \inf_{\mathbf{u}(\cdot)\in U} H\left(\mathbf{x}, \mathbf{u}, \frac{\partial}{\partial \mathbf{x}}V(\mathbf{x})\right) = 0 \tag{11}$$

where H is the Hamiltonian function. For the infinite-time LQR problem, the equation governing the value function is given by

$$\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\left[\mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}\right] + \frac{1}{2}\left[\mathbf{x}^T\mathbf{Q}(\mathbf{x})\mathbf{x} + \mathbf{u}^T\mathbf{R}(\mathbf{x})\mathbf{u}\right] = 0 \tag{12}$$

where the Hamiltonian is given by

$$H = \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\left[\mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}\right] + \frac{1}{2}\left[\mathbf{x}^T\mathbf{Q}(\mathbf{x})\mathbf{x} + \mathbf{u}^T\mathbf{R}(\mathbf{x})\mathbf{u}\right] \tag{13}$$

The boundary condition to (12) is given by $V(\mathbf{0}) = 0$ since in infinite-time the state goes to the origin. Stationary solutions of (12) are related to the stable Lagrangian manifolds of the Hamiltonian dynamics of the system, describing the behavior of the state, $\mathbf{x}$, and the adjoint variable, $\boldsymbol{\lambda}$,

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \boldsymbol{\lambda}} \tag{14}$$

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}} \tag{15}$$

Note that the origin of the Hamiltonian dynamics of the system yield a hyperbolic equilibrium at the origin.

Assumption 1. *The linearization of the system dynamics and the infinite-time performance index, (1) and (2), about the equilibrium is stabilizable and detectable (i.e.,* $\left\{\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{0}), \mathbf{B}(\mathbf{0}), \mathbf{Q}^{1/2}(\mathbf{0})\right\}$ *is stabilizable and detectable).*

Lemma 1. *Under Assumption 1, the equilibrium is hyperbolic and there exists a stable Lagrangian manifold, L, for the Hamiltonian dynamics that correspond to the dynamical system given by (1).*[17]

Since Assumption 1 can be used to construct a smooth value function $V(\mathbf{x})$ in the neighborhood of the origin, Lemma 1 implies the existence of a stable Lagrangian manifold, L, that goes through the origin. In fact, the value function, $V(\mathbf{x})$, is the generating function for the manifold (*i.e.*, L is the set of points $(\mathbf{x}, \boldsymbol{\lambda})$ satisfying $\boldsymbol{\lambda} = \partial V/\partial \mathbf{x}$). It can further be shown that the optimal feedback control is given by[16]

$$\mathbf{u}(\mathbf{x}) = -\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\left(\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\right)^T \tag{16}$$

Assumption 1 ensures the locally smooth optimal solution $V(\mathbf{x})$ near the origin; however when the optimal trajectories start to cross, this assumption begins to break down and a viscosity solution to the HJB equation, (12), is required. For a viscosity solution to exist, Assumption 2 must hold.[18]

Assumption 2. *The value function $V(\mathbf{x})$ satisfying the HJB equation, (12), is locally Lipschitz in a region Ω around the origin*

With existence in a region around the origin guaranteed (under the assumptions), the optimal feedback control, (16), can be substituted into (12), yielding

$$\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) - \frac{1}{2}\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\left(\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\right)^T + \frac{1}{2}\mathbf{x}^T\mathbf{Q}(\mathbf{x})\mathbf{x} = 0 \tag{17}$$

Van der Schaft[16] has shown, that since $\partial V(\mathbf{0})/\partial\mathbf{x} = \mathbf{0}$ that

$$\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} = \mathbf{P}(\mathbf{x})\mathbf{x} \tag{18}$$

for some matrix $\mathbf{P}(\mathbf{x}) = \mathbf{P}^T(\mathbf{x})$. Furthermore, van der Schaft showed that the factorization of $\mathbf{f}(\mathbf{x})$ into

$$\mathbf{f}(\mathbf{x}) = \mathbf{A}(\mathbf{x})\mathbf{x} \tag{19}$$

exists, provided the conditions given previously for (5) hold. Hence, the HJB equation given by (17) becomes

$$\mathbf{x}^T[\mathbf{P}(\mathbf{x})\mathbf{A}(\mathbf{x}) + \mathbf{A}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) - \mathbf{P}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) + \mathbf{Q}(\mathbf{x})]\mathbf{x} = 0 \tag{20}$$

where the symmetric nature of $\mathbf{P}(\mathbf{x})$ is utilized. The ARE is recovered directly from (20) setting the bracketed expression to zero for the linear case; however, since $\mathbf{P}(\mathbf{x})$ is required to be a gradient function to solve the HJB equation, this cannot be done for the nonlinear case. However, by relaxing the requirement that $\mathbf{P}(\mathbf{x})$ be a gradient function, (18), and instead require it be symmetric and positive-definite, a solution can be obtained by setting the bracketed expression in (20) to zero (notice that this is identical to solving the SDRE identified in (6)).

E. Existence of a Control Solution

The conditions required for the SDRE gain matrix to exist that results in the closed-loop SDC matrix, *i.e.*,

$$\mathbf{A_{CL}}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{K}(\mathbf{x}) \tag{21}$$

to be pointwise Hurwitz were derived by Cloutier, Stansbery, and Sznaier.[11]

Definition 1. The extended linearization of the system dynamics, (5), is a stabilizable (controllable) parameterization of the nonlinear system, (1), in a region $\Omega \in \mathbb{R}^n$ if $\{\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x})\}$ is pointwise stabilizable (controllable) in the linear sense for all $\mathbf{x} \in \Omega$.

Definition 2. The extended linearization of the system dynamics, (5), is a detectable (observable) parameterization of the nonlinear system, (1), in a region $\Omega \in \mathbb{R}^n$ if $\{\mathbf{A}(\mathbf{x}), \mathbf{Q}^{1/2}(\mathbf{x})\}$ is pointwise detectable (observable) in the linear sense for all $\mathbf{x} \in \Omega$.

Definition 3. The extended linearization of the system dynamics, (5), is pointwise Hurwitz in a region Ω if the eigenvalues of $\mathbf{A}(\mathbf{x})$ lie in the open left half of the complex plane (*i.e.*, Re(λ)< 0) for all $\mathbf{x} \in \Omega$.

Definition 4. A $\mathcal{C}^1$ control law, (3), is recoverable by SDRE control in a region Ω if there exists a pointwise stabilizable SDC parameterization $\{\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x})\}$, a pointwise positive-semidefinite state weighting matrix $\mathbf{Q}(\mathbf{x})$, and a pointwise positive-definite control weighting matrix $\mathbf{R}(\mathbf{x})$ such that resulting state-dependent controller, (7), satisfies the general regulation control law formulation, (3), for all $\mathbf{x}$.

With these definitions in place, it has been shown that if Theorem 1 holds, then a SDRE gain matrix exists that renders in the closed-loop SDC matrix pointwise Hurwitz.[11]

Theorem 1. *A $\mathcal{C}^1$ control law, (3), is recoverable by SDRE control in a region Ω if there exists a pointwise stabilizable SDC parameterization $\{\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x})\}$ such that the closed-loop dynamics matrix (21) is pointwise Hurwitz in Ω, and the gain $\mathbf{K}(\mathbf{x})$ satisfies the pointwise minimum-phase property in Ω, that is, the zeros of the loop gain $\mathbf{K}(\mathbf{x})\left[s\mathbf{I} - \mathbf{A}(\mathbf{x})\right]^{-\mathbf{1}}\mathbf{B}(\mathbf{x})$ lie in the closed left have plane Re(s) ≤ 0, pointwise.*

F. Stability of the Closed-Loop SDRE System

In order to guarantee local asymptotic stability, Mracek and Cloutier,[8] first made the following assumptions

Assumption 3. *$\mathbf{A}(\cdot)$, $\mathbf{B}(\cdot)$, $\mathbf{Q}(\cdot)$, and $\mathbf{R}(\cdot)$ are $\mathcal{C}^1$ matrix-valued functions.*

Assumption 4. *The pairs $\{\mathbf{A}(\mathbf{x}),\mathbf{B}(\mathbf{x})\}$ and $\{\mathbf{A}(\mathbf{x}),\mathbf{Q}^{1/2}(\mathbf{x})\}$ are pointwise stabilizable and detectable parameterizations of the nonlinear system, (1), for all $\mathbf{x}$.*

They then prove that the following theorem must hold

Theorem 2. *The nonlinear system, (1), with feedback control determined by the SDRE, (7), is locally asymptotically stable provided that*

(i) $\mathbf{x} \in \mathbb{R}^n$, *($n > 1$)*

(ii) $\mathbf{P}(\mathbf{x})$ *is the unique, symmetric, positive-definite, pointwise-stabilizing solution of the SDRE, (6)*

(iii) Assumptions 3 and 4 hold

Proof. Using SDRE control, the closed-loop system is given by $\dot{\mathbf{x}} = \mathbf{A_{CL}}(\mathbf{x})\mathbf{x}$, where the closed-loop SDC matrix, $\mathbf{A_{CL}}(\mathbf{x})$, is given by (21). Under Assumption 3, $\mathbf{P}(\mathbf{x})$ is $\mathcal{C}^1$, and so is $\mathbf{A_{CL}}(\mathbf{x})$. The Mean Value Theorem applied to the closed-loop SDC matrix gives

$$\mathbf{A_{CL}}(\mathbf{x}) = \mathbf{A_{CL}}(\mathbf{0})\mathbf{x} + \frac{\partial \mathbf{A_{CL}}(\mathbf{z})}{\partial \mathbf{x}}\mathbf{x}$$

where $\mathbf{z}$ is a vector on the line segment joining the origin and $\mathbf{x}$. Substituting this expression into the closed-loop system dynamics gives

$$\dot{\mathbf{x}} = \mathbf{A_{CL}}(\mathbf{0}) + \mathbf{x}^T \frac{\partial \mathbf{A_{CL}}(\mathbf{z})}{\partial \mathbf{x}}\mathbf{x}$$

and

$$\dot{\mathbf{x}} = \mathbf{A_{CL}}(\mathbf{0}) + \mathbf{\Psi}(\mathbf{x},\mathbf{z}) \parallel \mathbf{x} \parallel$$

where

$$\mathbf{\Psi}(\mathbf{x},\mathbf{z}) \triangleq \frac{1}{\parallel \mathbf{x} \parallel}\mathbf{x}^T \frac{\partial \mathbf{A_{CL}}(\mathbf{z})}{\partial \mathbf{x}}\mathbf{x}$$

This definition implies that as $\parallel \mathbf{x} \parallel \rightarrow \infty$ then $\mathbf{\Psi}(\mathbf{x},\mathbf{z}) \rightarrow \mathbf{0}$. Therefore, there exists a neighborhood around the origin where the linear term, $\mathbf{A_{CL}}(\mathbf{0})$ dominates the dynamics, and local asymptotic stability is ensured by this matrix being Hurwitz, as was previously shown. □

Hence, Theorem 2 provides conditions that guarantee the local asymptotic stability of the closed-loop nonlinear dynamical system. Furthermore, these conditions are fairly readily achievable for a large class of extended linearized systems.

Global asymptotic stability of the closed-loop system is, in general, harder to achieve, as this implies that the system is stable for *any* initial conditions. In order to achieve global stability, it is not sufficient to prove that the eigenvalues of $\mathbf{A_{CL}}(\mathbf{x})$ have negative real parts for all $\mathbf{x}$, as this is an extended linearization of the nonlinear system. For the global asymptotic stability of the general multivariable case consider Theorem 3, derived by Cloutier, D'Souza, and Mracek.[7]

Theorem 3. *If the closed-loop SDC matrix, (21), is symmetric for all $\mathbf{x}$, then under Assumptions 3 and 4, the SDRE closed-loop control law, (7), renders the closed-loop nonlinear system globally asymptotically stable.*

Proof. Let $V(\mathbf{x}) = \mathbf{x}^T\mathbf{x}$ be a candidate Lyapunov function. Then

$$\dot{V}(\mathbf{x}) = \mathbf{x}^T\dot{\mathbf{x}} + \dot{\mathbf{x}}^T\mathbf{x} = \mathbf{x}^T \left[\mathbf{A_{CL}}(\mathbf{x}) + \mathbf{A_{CL}}^T(\mathbf{x})\right]\mathbf{x}$$

Under Assumptions 3 and 4, the closed-loop SDC matrix, (21), is stable for all $\mathbf{x}$. If the closed-loop SDC matrix is symmetric, then the quantity $\mathbf{A_{CL}}(\mathbf{x}) + \mathbf{A_{CL}}^T(\mathbf{x}) < 0$, and $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x}$. □

Clearly, the restrictions of Theorem 3 are restrictive and difficult to ensure in the general case. Hence, for most systems, local asymptotic stability is assured and a region of attraction can be estimated.

G. Estimation of the Region of Attraction

McCaffrey and Banks[19] proposed a method for estimating the region of attraction of the closed-loop system resulting from SDRE control, that is, the region in state space that encloses all initial conditions such that the origin of the system is reached asymptotically. Their methodology invokes the Hamiltonian dynamics, (14) and (15), and the Lagrangian manifold.

Proposition 1. *For any $t > 0$ such that $\Omega_t \subset \Omega$, $V(\mathbf{x})$ is strictly decreasing along trajectories of the closed-loop system, (9), for all $\mathbf{x_0} \in \Omega_t \setminus \{0\}$ provided*

$$\frac{1}{2}[\boldsymbol{\lambda} - \mathbf{P}(\mathbf{x})\mathbf{x}]^T\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})[\boldsymbol{\lambda} - \mathbf{P}(\mathbf{x})\mathbf{x}] - \frac{1}{2}\mathbf{x}^T\mathbf{P}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x})\mathbf{x} \leq 0 \tag{22}$$

for all $(\mathbf{x}, \boldsymbol{\lambda}) \in L$ such that $\mathbf{x} \in \Omega_t \setminus \{0\}$.

Noting that $\mathbf{P}(\mathbf{0}) = \frac{\partial^2 V(\mathbf{0})}{\partial \mathbf{x}^2}$ so that as $\mathbf{x} \to \mathbf{0}$, $\mathbf{P}(\mathbf{x})\mathbf{x} \to \frac{\partial^2 V(\mathbf{0})}{\partial \mathbf{x}^2}\mathbf{x}$ and $\boldsymbol{\lambda} \to \frac{\partial^2 V(\mathbf{0})}{\partial \mathbf{x}^2}\mathbf{x}$. Therefore, $\mathbf{P}(\mathbf{x})\mathbf{x} \to \boldsymbol{\lambda}$ as $\mathbf{x} \to \mathbf{0}$, and the proposition, (22), will hold in a ball $\mathcal{B}_\varepsilon$ around the origin, where ε is arbitrarily small. With this framework in place, the stability region can be estimated by:

1. Integrating trajectories of the Hamiltonian dynamics, (14) and (15), backwards in time from $\mathbf{x}_f \in \partial\mathcal{B}_\varepsilon$ and $\boldsymbol{\lambda}_f = \frac{\partial^2 V(\mathbf{0})}{\partial \mathbf{x}^2}\mathbf{x}_f$
2. Estimate the largest t for which (22) holds in Ω_t

Upon finding this time, the region of attraction is the sublevel set defined by $\{\mathbf{x} \in \mathbb{R}^n : V(\mathbf{x}) \leq \beta\}$, where $\beta = \min\{V(\mathbf{x}) : \mathbf{x} \in \partial\Omega_t\}$.

H. Optimality of the SDRE

As $\mathbf{x} \to \mathbf{0}$, $\mathbf{A}(\mathbf{x}) \to \partial\mathbf{f}(\mathbf{0})/\partial\mathbf{x}$ which implies that $\mathbf{P}(\mathbf{x})$ approaches the linear ARE at the origin. Furthermore, the SDRE control solution asymptotically approaches the optimal control as $\mathbf{x} \to \mathbf{0}$ and away from the origin the SDRE control is arbitrarily close to the optimal feedback. Hence the SDRE approach yields an asymptotically optimal feedback solution. Mracek and Cloutier[8] developed the necessary conditions for the optimality of a general nonlinear regulator, that is the regulator governed by (1) and (2), and then extend those results to determine the optimality of the SDRE approach.

Assumption 5. *$\mathbf{A}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$, $\mathbf{P}(\mathbf{x})$, $\mathbf{Q}(\mathbf{x})$, and $\mathbf{R}(\mathbf{x})$ and their respective gradients are bounded in a neighborhood Ω of the origin.*

Theorem 4. *For the general multivariable nonlinear SDRE control case (i.e., $n > 1$), the SDRE nonlinear feedback solution and its associated state and costate trajectories satisfy the first necessary condition for optimality (i.e., $\partial H/\partial\mathbf{u} = \mathbf{0}$ of the nonlinear optimal regulator problem defined by (1) and (2). Additionally, if Assumptions 4 and 5 hold under asymptotic stability, as $\mathbf{x} \to \mathbf{0}$, the second necessary condition for optimality (i.e., $\dot{\boldsymbol{\lambda}} = -\partial H/\partial\mathbf{x}$) is asymptotically satisfied at a quadratic rate.*

Proof. Pontryagin's maximum principle states that necessary conditions for optimality are

$$\frac{\partial H}{\partial \mathbf{u}} = \mathbf{0}, \qquad \dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}}, \qquad \dot{\mathbf{x}} = \frac{\partial H}{\partial \boldsymbol{\lambda}} \tag{23}$$

where H is the Hamiltonian, (13), and $\boldsymbol{\lambda} = \partial H/\partial\mathbf{x}$. Using these definitions and (7) yield

$$\frac{\partial H}{\partial \mathbf{u}} = \mathbf{B}^T(\mathbf{x})[\boldsymbol{\lambda} - \mathbf{P}(\mathbf{x})\mathbf{x}] \tag{24}$$

Furthermore, since $\boldsymbol{\lambda} = \partial V/\partial\mathbf{x}$ the adjoint vector for the system satisfies

$$\boldsymbol{\lambda} = \mathbf{P}(\mathbf{x})\mathbf{x} \tag{25}$$

and the first optimality condition, (24) is satisfied identically for the nonlinear regulator problem.
With the Hamiltonian defined in (13), the second necessary condition becomes

$$\dot{\boldsymbol{\lambda}} = -\left(\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right)^T \boldsymbol{\lambda} - \mathbf{u}^T\left(\frac{\partial \mathbf{B}(\mathbf{x})}{\partial \mathbf{x}}\right)^T \boldsymbol{\lambda} - \mathbf{Q}(\mathbf{x})\mathbf{x} - \frac{1}{2}\mathbf{x}^T\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{x} - \frac{1}{2}\mathbf{u}^T\frac{\partial \mathbf{R}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{u} \tag{26}$$

Taking the time derivative of (25) yields

$$\dot{\boldsymbol{\lambda}} = \dot{\mathbf{P}}(\mathbf{x})\mathbf{x} + \mathbf{P}(\mathbf{x})\dot{\mathbf{x}} \tag{27}$$

Substituting this result, along with (7), (9), (19), and (26) into (6) and rearranging yield the *SDRE Necessary Condition for Optimality*[8]

$$\begin{aligned}\dot{\mathbf{P}}(\mathbf{x})\mathbf{x} + \frac{1}{2}\mathbf{x}^T\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{P}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\frac{\partial \mathbf{R}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x})\mathbf{x} \\ + \mathbf{x}^T\left(\frac{\partial \mathbf{A}(\mathbf{x})}{\partial \mathbf{x}}\right)^T\mathbf{P}(\mathbf{x})\mathbf{x} - \mathbf{x}^T\mathbf{P}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\left(\frac{\partial \mathbf{B}(\mathbf{x})}{\partial \mathbf{x}}\right)^T\mathbf{P}(\mathbf{x})\mathbf{x} = \mathbf{0}\end{aligned} \tag{28}$$

Hence, whenever (28) is satisfied, the closed-loop SDRE solution satisfies all the first-order necessary conditions for an extremum of the cost functional.

In general, (28) is not satisfied for a given extended linearization of the nonlinear system; however, the suboptimality can be examined as

$$\dot{\mathbf{P}}(\mathbf{x})\mathbf{x} = \left(\sum_{i=1}^{n}\frac{\partial \mathbf{P}(\mathbf{x})}{\partial x_i}\dot{x}_i\right) = \sum_{i=1}^{n}\frac{\partial \mathbf{P}(\mathbf{x})}{\partial x_i}\left[a_{CL}^i\mathbf{x}\right]\mathbf{x} \tag{29}$$

where a_{CL}^i is the i^{th} row of the closed-loop SDC matrix. Substituting (29) into (28) yields the condition

$$\mathbf{x}^T\mathbf{M}_i\mathbf{x} = \mathbf{0} \tag{30}$$

where

$$\begin{aligned}\mathbf{M}_i \triangleq & \mathbf{N}_i + \frac{1}{2}\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial x_i} + \frac{1}{2}\mathbf{P}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\frac{\partial \mathbf{R}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) + \left(\frac{\partial \mathbf{A}(\mathbf{x})}{\partial \mathbf{x}}\right)^T\mathbf{P}(\mathbf{x})\mathbf{x} \\ & - \mathbf{x}^T\mathbf{P}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\left(\frac{\partial \mathbf{B}(\mathbf{x})}{\partial \mathbf{x}}\right)^T\mathbf{P}(\mathbf{x})\end{aligned} \tag{31}$$

where $\mathbf{N}_i$ is defined from the relationship

$$\sum_{i=1}^{n}\frac{\partial \mathbf{P}(\mathbf{x})}{\partial x_i}\left[a_{CL}^i\mathbf{x}\right]\mathbf{x} = \mathbf{x}^T\mathbf{N}_i\mathbf{x} \tag{32}$$

With asymptotic stability, the trajectories will eventually enter and remain in Ω and under Assumption 5, there exists a constant, positive definite matrix $\mathbf{U}$ such that

$$\max_i \mid \mathbf{x}^T\mathbf{M}_i\mathbf{x} \mid \leq \mathbf{x}^T\mathbf{U}\mathbf{x} \qquad \forall \mathbf{x} \in \Omega \tag{33}$$

Hence, the ∞-norm of the SDRE Necessary Condition, (28), is bounded by a quadratic, positive-definite function from above, and so the suboptimality of the solution is bounded from above. □

IV. Sum-of-Squares Analysis and Control Synthesis Techniques

A. Sum-of-Squares Decomposition

A multivariate polynomial, $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$ is said to be a sum-of-squares if there exist polynomials $f_1(\mathbf{x})$,..., $f_m(\mathbf{x})$ such that

$$f(\mathbf{x}) = \sum_{i=1}^{m} f_i^2(\mathbf{x}) \tag{34}$$

This statement is equivalent to the following proposition.[12]

Proposition 2. *Let $f(\mathbf{x})$ be a polynomial in $\mathbf{x} \in \mathbb{R}^n$ of degree $2d$. In addition, let $\mathbf{Z}(\mathbf{x})$ be a column vector whose entries are all monomials in $\mathbf{x}$ with degree no greater than d. Then $f(\mathbf{x})$ is a sum-of-squares if and only if there exists a positive semi-definite matrix $\mathbf{Q}$ such that*

$$f(\mathbf{x}) = \mathbf{Z}^T(\mathbf{x})\mathbf{Q}\mathbf{Z}(\mathbf{x}) \tag{35}$$

With this definition, it can be seen that a sum-of-squares decomposition can be found using semidefinite programming, to search for the $\mathbf{Q}$ matrix satisfying (35).

What is significant about sum-of-squares decomposition for control applications, is that when the polynomial $f(\mathbf{x})$ has coefficients that are parameterized in terms of some other unknowns, but the polynomial is yet unknown. A search for the coefficients that render the polynomial $f(\mathbf{x})$ a sum-of-squares can still be performed using semidefinite programming. For example, consider the construction of a Lyapunov function for a nonlinear system where the following procedure can be used.

1. Coefficients can be used to parameterize a set of candidate Lyapunov functions in an affine manner, that is it can determine a set $\mathcal{V} = \{V(\mathbf{x}) : V(\mathbf{x}) = v_0(\mathbf{x}) + \sum_{i=1}^{m} c_i v_i(\mathbf{x})\}$, where the $v_i(\mathbf{x})$'s are monomials in $\mathbf{x}$.
2. Search for a function $V(x) \in \mathcal{V}$ which satisfies $V(\mathbf{x}) - \phi(\mathbf{x})$ and $-\dfrac{\partial V(\mathbf{x})}{\partial \mathbf{x}} f(\mathbf{x})$, where $\phi(\mathbf{x}) > 0$ using semidefinite programing

The semidefinite programming problem above determines the state dependent linear matrix inequalities (LMIs) that govern the problem and is a resultants of solving the the following convex optimization problem

$$\text{Minimize:} \quad \sum_{i=1}^{m} a_i c_i \tag{36}$$

$$\text{Subject to:} \quad \mathbf{F}_0(\mathbf{x}) + \sum_{i=1}^{m} c_i \mathbf{F}_i(\mathbf{x}) \geq 0 \tag{37}$$

where $a_i \in \mathbb{R}$ are fixed coefficients, $c_i \in \mathbb{R}$ are decision variables, and $\mathbf{F}_i(\mathbf{x})$ are symmetric matrix functions of the indeterminate $\mathbf{x} \in \mathbb{R}^n$. When $\mathbf{F}_i(\mathbf{x})$ are symmetric polynomial matrices in $\mathbf{x}$ the computationally difficult problem of solving (36) and (37) is relaxed according to the following proposition[12]

Proposition 3. *Let $\mathbf{F}(\mathbf{x})$ be an $m \times m$ symmetric polynomial matrix of degree $2d$ in $\mathbf{x} \in \mathbb{R}^n$. Furthermore, let $\mathbf{Z}(\mathbf{x})$ be a column vector whose entries are all monomials in $\mathbf{x}$ with degree no greater than d, and assume the following:*

(i) $\mathbf{F}(\mathbf{x}) \geq 0 \quad \forall \mathbf{x} \in \mathbb{R}^n$

(ii) $\mathbf{v}^T \mathbf{F}(\mathbf{x}) \mathbf{v}$ *is a sum of squares, with* $\mathbf{v} \in \mathbb{R}^m$

(iii) There exists a positive semi-definite matrix $\mathbf{Q}$ *such that* $\mathbf{v}^T \mathbf{F}(\mathbf{x}) \mathbf{v} = (\mathbf{v} \otimes \mathbf{Z}(\mathbf{x}))^T \mathbf{Q} (\mathbf{v} \otimes \mathbf{Z}(\mathbf{x}))$

Then (i) $\Leftarrow$ (ii) and (ii) $\Leftrightarrow$ (iii)

This proposition is proven by Prajna, Papachristodoulou, and Wu in.[12] However, by applying Proposition 3, it is seen that the solution to the sum-of-squares optimization problem seen in (38) and (39) is also a solution to the state-dependent LMI problem, (36) and (37).

$$\text{Minimize:} \quad \sum_{i=1}^{m} a_i c_i \tag{38}$$

$$\text{Subject to:} \quad \mathbf{v}^T \left(\mathbf{F}_0(\mathbf{x}) + \sum_{i=1}^{m} c_i \mathbf{F}_i(\mathbf{x}) \right) \mathbf{v} \quad \text{is a sum-of-squares polynomial} \tag{39}$$

This relaxation of the LMI problem turns the relatively difficult computation problem associated with (36) and (37) to a relatively simple computational problem since semidefinite programming solvers are readily available on multiple platforms,[13].[14]

B. State-Feedback Control Synthesis Using Sum-of-Squares

Considering the dynamical system given by (1) which can be rewritten similarly to the extended linearized equation, (5) with the modification that the state SDC matrix now multiplies a monomial vector of the state. That is

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x})\mathbf{Z}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u} \tag{40}$$

where now $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are polynomial matrices in $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$ and $\mathbf{Z}(\mathbf{x})$ is an $n \times 1$ vector of monomials in $\mathbf{x}$ satisfying

Assumption 6. $\mathbf{Z}(\mathbf{x}) = \mathbf{0}$ *if and only if* $\mathbf{x} = \mathbf{0}$

Furthermore, define a matrix $\mathbf{M}(\mathbf{x})$ to be an $n \times m$ polynomial matrix given by the relationship

$$M_{ij}(\mathbf{x}) = \frac{\partial Z_i}{\partial x_j}(\mathbf{x}) \quad i = 1, ..., n, \quad j = 1, ..., m \tag{41}$$

Also, let $\mathbf{A}_j(\mathbf{x})$ denote the j^{th} row of $\mathbf{A}(\mathbf{x})$, $J = \{j_1, j_2, ..., j_m\}$ denote the row indicies of $\mathbf{B}(\mathbf{x})$ whose corresponding row is equal to zero, and define $\bar{\mathbf{x}} = (x_{j1}, x_{j2}, ..., x_{jm})$. It is desired to find a state-feedback control law of the form $\mathbf{K}(\mathbf{x}) = \mathbf{F}(\mathbf{x})\mathbf{Z}(\mathbf{x})$ that renders the equilibrium stable. Consider the following lemma[12]

Lemma 2. *For a symmetric polynomial matrix,* $\mathbf{P}(\mathbf{x})$*, that is nonsingular* $\forall \mathbf{x} \in \mathbb{R}^n$*,*

$$\frac{\partial \mathbf{P}(\mathbf{x})}{\partial x_i} = -\mathbf{P}(\mathbf{x})\frac{\partial \mathbf{P}^{-1}(\mathbf{x})}{\partial x_i}\mathbf{P}(\mathbf{x}) \tag{42}$$

Proof. Since $\mathbf{P}(\mathbf{x})$ is non-singular, $\mathbf{P}^{-1}(\mathbf{x})\mathbf{P}(\mathbf{x}) = \mathbf{I}$. Taking the partial of both sides with respect to x_i gives

$$\frac{\partial \mathbf{P}(\mathbf{x})}{\partial x_i}\mathbf{P}^{-1}(\mathbf{x}) + \mathbf{P}(\mathbf{x})\frac{\partial \mathbf{P}^{-1}(\mathbf{x})}{\partial x_i} = \mathbf{0}$$

which when rearranged is precisely (42) □

The following theorem then guarantees the existence of feedback control law that stabilizes (40)[12]

Theorem 5. *For a dynamical system of the form (40), suppose there exists an* $n \times n$ *symmetric polynomial matrix* $\mathbf{P}(\bar{\mathbf{x}})$*, an* $m \times n$ *polynomial matrix* $\mathbf{K}(\mathbf{x})$*, a constant* $\varepsilon_1 > 0$*, and a sum-of-squares polynomial* $\varepsilon_2(\mathbf{x})$*, such that*

$$\mathbf{v}^T \left(\mathbf{P}(\bar{\mathbf{x}}) - \varepsilon_1 \mathbf{I}\right) \mathbf{v} \tag{43}$$

and

$$\begin{aligned} -\mathbf{v}^T (&\mathbf{P}(\bar{\mathbf{x}})\mathbf{A}^T(\mathbf{x})\mathbf{M}^T(\mathbf{x}) + \mathbf{M}(\mathbf{x})\mathbf{A}(\mathbf{x})\mathbf{P}(\bar{\mathbf{x}}) + \mathbf{K}^T(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{M}^T(\mathbf{x}) + \mathbf{M}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{K}(\mathbf{x}) \\ &- \sum_{j \in J} \frac{\partial \mathbf{P}(\bar{\mathbf{x}})}{\partial x_j}\left(\mathbf{A}_j(\mathbf{x})\mathbf{Z}(\mathbf{x})\right) + \varepsilon_2(\mathbf{x})\mathbf{I})\mathbf{v} \end{aligned} \tag{44}$$

are sums-of-squares, with $\mathbf{v} \in \mathbb{R}^n$*. Then the state-feedback stabilization problem is solvable with a controller given by*

$$\mathbf{u}(\mathbf{x}) = \mathbf{K}(\mathbf{x})\mathbf{P}^{-1}(\bar{\mathbf{x}})\mathbf{Z}(\mathbf{x}) \tag{45}$$

Furthermore, if (44) holds with $\varepsilon_2(\mathbf{x}) > 0 \quad \forall \mathbf{x} \setminus \mathbf{0}$*, then the origin is asymptotically stable, and if* $\mathbf{P}(\bar{\mathbf{x}})$ *is a constant matrix, then the origin is globally asymptotically stable.*

Proof. Assume that there exist solutions $\mathbf{P}(\bar{\mathbf{x}})$ and $\mathbf{K}(\mathbf{x})$ to (43) and (44). Define a Lyapunov function candidate as

$$V(\mathbf{x}) = \mathbf{Z}(\mathbf{x})^T\mathbf{P}^{-1}(\bar{\mathbf{x}})\mathbf{Z}(\mathbf{x})$$

for the closed-loop system

$$\dot{\mathbf{x}} = \left[\mathbf{A}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{K}(\mathbf{x})\mathbf{P}^{-1}(\bar{\mathbf{x}})\right]\mathbf{Z}(\mathbf{x})$$

Using Proposition 3, (43) implies that $\mathbf{P}(\bar{\mathbf{x}})$ and $\mathbf{P}^{-1}(\bar{\mathbf{x}})$ are positive definite for all $\mathbf{x}$, and $V(\mathbf{x}) > 0 \quad \forall \mathbf{x} \in \mathbb{R}^n$. Looking at the derivative of the Lyapunov function candidate along trajectories of the system

$$\begin{aligned}\dot{V}(\mathbf{x}(t)) =& \mathbf{Z}^T(\mathbf{x})[\sum_{j\in J} \frac{\partial \mathbf{P}^{-1}(\bar{\mathbf{x}})}{\partial x_j} (\mathbf{A}_j(\mathbf{x})\mathbf{Z}(\mathbf{x})) + \left[\mathbf{A}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{K}(\mathbf{x})\mathbf{P}^{-1}(\bar{\mathbf{x}})\right]^T \mathbf{M}^T(\mathbf{x})\mathbf{M}(\mathbf{x}) \\ &+ \mathbf{P}^{-1}(\bar{\mathbf{x}})\mathbf{M}(\mathbf{x}) \left[\mathbf{A}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{K}(\mathbf{x})\mathbf{P}^{-1}(\bar{\mathbf{x}})\right]]\mathbf{Z}(\mathbf{x})\end{aligned}$$

Using (44), the expression

$$\mathbf{P}(\bar{\mathbf{x}})\mathbf{A}^T(\mathbf{x})\mathbf{M}^T(\mathbf{x}) + \mathbf{M}(\mathbf{x})\mathbf{A}(\mathbf{x})\mathbf{P}(\bar{\mathbf{x}}) + \mathbf{K}^T(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{M}^T(\mathbf{x}) + \mathbf{M}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{K}(\mathbf{x}) - \sum_{j\in J} \frac{\partial \mathbf{P}(\bar{\mathbf{x}})}{\partial x_j} (\mathbf{A}_j(\mathbf{x})\mathbf{Z}(\mathbf{x}))$$

is negative semidefinite for all $\mathbf{x}$. Hence, using Lemma 2, $\dot{V}(\mathbf{x}(t)) \ngtr 0$ and the closed-loop system is stable. Furthermore, if (44) holds with $\varepsilon_2(\mathbf{x}) > 0 \quad \forall \mathbf{x} \setminus \mathbf{0}$, then $\dot{V}(\mathbf{x}(t)) < 0$ and the origin is asymptotically stable. Also, if $\mathbf{P}(\bar{\mathbf{x}})$ is a constant matrix, then $V(\mathbf{x})$ is radially unbounded, and the origin is globally asymptotically stable. □

C. Optimal Stabilization of a Nonlinear System

Consider the nonlinear, infinite time LQR problem previously discussed, in which the state is driven to the origin such that a performance index that is a function of state and control is extremalized. In terms of the sum-of-squares framework, the LQR problem is of the form

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{z_1} \\ \mathbf{z_2} \end{bmatrix} = \begin{bmatrix} \mathbf{A}(\mathbf{x}) & \mathbf{B}(\mathbf{x}) \\ \mathbf{C_1}(\mathbf{x}) & \mathbf{0} \\ \mathbf{C_2}(\mathbf{x}) & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{Z}(\mathbf{x}) \\ \mathbf{u} \end{bmatrix}, \quad \mathbf{x}(0) = \mathbf{x_0} \tag{46}$$

where $\mathbf{Z}(\mathbf{x})$ is a monomial vector satisfying Assumption 6 and $\mathbf{C_1}(\mathbf{x}) \neq \mathbf{0}$ when $\mathbf{x} \neq \mathbf{0}$. The performance index to be minimized is given by

$$J\left(\mathbf{x}(t), \mathbf{u}(t)\right) = \int_0^\infty (\mathbf{z_1}^T(t)\mathbf{z_1}(t) + \mathbf{z_2}^T(t)\mathbf{z_2}(t))dt \tag{47}$$

Notice that the this performance index is just $\| \mathbf{z} \|_2^2$. The following theorem governs the existence of the state-feedback control law[12]

Theorem 6. *Suppose that (46) there exists an $n \times n$ symmetric polynomial matrix $\mathbf{P}(\bar{\mathbf{x}})$, a constant $\varepsilon_1 > 0$, and a sum-of-squares polynomial $\varepsilon_2(\mathbf{x}) > 0 \quad \forall \mathbf{x} \neq \mathbf{0}$, such that*

$$\mathbf{v_1}^T \left(\mathbf{P}(\bar{\mathbf{x}}) - \varepsilon_1 \mathbf{I}\right) \mathbf{v_1} \tag{48}$$

and

$$-\left[\mathbf{v_1}^T \Gamma \mathbf{v_1} + \mathbf{v_1}^T \Lambda \mathbf{v_2} + \mathbf{v_2}^T \Delta \mathbf{v_1} + \mathbf{v_2}^T \Phi \mathbf{v_2}\right] \tag{49}$$

are sum-of-squares, where $\mathbf{v_1} \in \mathbb{R}^n$, $\mathbf{v_2} \in \mathbb{R}^m$, and

$$\Gamma = \mathbf{M}\hat{\mathbf{A}}\mathbf{P} + \mathbf{P}\hat{\mathbf{A}}^T\mathbf{M}^T - \mathbf{M}\mathbf{B}\mathbf{B}^T\mathbf{M}^T - \sum_{j\in J} \frac{\partial \mathbf{P}}{\partial x_j} (\mathbf{A}_j \mathbf{Z}) + \varepsilon_2 \mathbf{I} \tag{50}$$

$$\Lambda = \mathbf{P}\mathbf{C_1}^T \tag{51}$$

$$\Delta = \mathbf{C_1}\mathbf{P} \tag{52}$$

$$\Phi = -(1 - \varepsilon_2)\mathbf{I} \tag{53}$$

where in the previous equations the dependence on the state, $\mathbf{x}$, has been dropped for simplicity. Furthermore, $\hat{\mathbf{A}}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{C_2}(\mathbf{x})$. Then the state-feedback control law

$$\mathbf{u}(\mathbf{x}) = -\left[\mathbf{B}^T(\mathbf{x})\mathbf{M}^T(\mathbf{x})\mathbf{P}^{-1}(\bar{\mathbf{x}}) + \mathbf{C_2}(\mathbf{x})\right] \mathbf{Z}(\mathbf{x}) \tag{54}$$

causes the origin to be asymptotically stable, and if $P(\bar{\mathbf{x}})$ is a constant matrix, the origin is globally asymptotically stable. Additionally, for an initial condition, $\mathbf{x_0}$, that is inside the region of attraction the performance index is bounded by

$$J\left(\mathbf{x}(t), \mathbf{u}(t)\right) = \parallel \mathbf{z} \parallel_2^2 \leq \mathbf{Z}^T(\mathbf{x_0})\mathbf{P}^{-1}(\bar{\mathbf{x}}_\mathbf{0})\mathbf{Z}(\mathbf{x_0}) \tag{55}$$

The proof of this theorem follows similarly to the general sum-of-squares stabilization problem discussed in the previous section, and is outlined in.[12]

D. Connection Between the SDRE and Sum-of-Squares Framework

Note that the framework setup to solve the infinite time LQR problem for both the SDRE problem and sum-of-squares control synthesis problem can be made identical assuming that the monomial vector $\mathbf{Z}(\mathbf{x})$ is just the state vector, $\mathbf{x}$. That is to say,

$$\mathbf{Z}(\mathbf{x}) = \mathbf{x} \tag{56}$$

and the extended linearization of $\mathbf{f}(\mathbf{x})$ into the SDC matrix, $\mathbf{A}(\mathbf{x})$, results in the SDC matrix being composed entirely of polynomial functions. With these restrictions in place, the state-dependent LMI solution solved by the SDRE yields the same solution and the solution obtained by sum-of-squares control synthesis.

V. Dynamics of a High Mass Mars Entry Vehicle

For this investigation, the nonlinear equations of motion shown in (57)-(64) will govern the flight mechanics of the vehicle during its descent through the atmosphere. Note that these equations of motion assume three degree-of-freedom motion of a point mass with rotation about the velocity vector (*i.e.*, bank) being decoupled from any other rotational motion of the vehicle. Additionally, because the hypersonic flight time for a vehicle descending through the Martian atmosphere is relatively short relative to the rotational period, the effect of planetary rotation is ignored.

$$\dot{\theta} = \frac{V}{r}\frac{\cos(\gamma)\cos(\psi)}{\cos(\phi)} \tag{57}$$

$$\dot{\phi} = \frac{V}{r}\cos(\gamma)\sin(\psi) \tag{58}$$

$$\dot{r} = -V\sin(\gamma) \tag{59}$$

$$\dot{\psi} = -\frac{V}{r}\cos(\gamma)\cos(\psi)\tan(\phi) + \frac{\hat{L}}{V}\cos(\gamma)\sin(\sigma) \tag{60}$$

$$\dot{\gamma} = \left(g - \frac{V^2}{r}\right)\frac{\cos(\gamma)}{V} - \frac{\hat{L}}{V}\cos(\sigma) \tag{61}$$

$$\dot{V} = -\hat{D} + g\sin(\gamma) \tag{62}$$

$$\ddot{\sigma} = \frac{Td}{I_b} \tag{63}$$

$$\dot{m} = -\frac{T}{g_0 I_{sp}} \tag{64}$$

Where (57)-(59) are the kinematic equations governing the time rate of change of longitude (θ), latitude (ϕ), and the radial distance from the center of the planet (r). Equations (60)-(62) are the equations of motion describing the time rate of change of the azimuth (ψ), flight path angle (γ), and velocity (V). The last two equations above, (63) and (64), describe the attitude dynamics of the vehicle and the associated vehicle mass loss with using the thrusters. While the bank angle (σ) is the control variable that appears directly in the dynamics of the vehicle (57)-(62), the thrust (T) is the actuator available to control the bank angle. The relationships for the mass specific lift ($\hat{L}$) and mass specific drag ($\hat{D}$) are seen in (65) and (66).

$$\hat{L} = \frac{1}{2m}\rho V^2 S C_L \tag{65}$$

$$\hat{D} = \frac{1}{2m}\rho V^2 S C_D \tag{66}$$

Additionally, the planetary environment is assumed to be defined by an inverse-square law gravitational field and an exponential density profile, as seen in (67) and (68)

$$g = g_p \left(\frac{r_p}{r}\right)^2 \tag{67}$$

$$\rho = \rho_p \exp\left(-\frac{r - r_p}{H}\right) \tag{68}$$

The remaining parameters of (57)-(68) are shown in Table 1 where values specific to Mars and the Mars Science Laboratory vehicle are used.

Table 1. Additional Modeling Parameters

Parameter	Description	Value	Units
g_0	Earth Reference Gravity	9.806	m/s^2
g_p	Mars Surface Gravity	3.71	m/s^2
r_p	Mars Surface Radius	3397	km
ρ_p	Mars Surface Density	0.0068	kg/m^3
H	Atmospheric Scale Height	17391	m
d	Thruster Offset Distance	0.9	m
I_{sp}	Specific Impulse of Thruster	190	s
m_0	Entry Mass	2196	kg
I_b	Vehicle Moment of Inertia	5560	kg-m^2
S	Vehicle Reference Area	15.9	m^2
C_D	Coefficient of Drag	1.4	–
C_L	Coefficient of Lift	0.34	–

VI. Modified Apollo Final Phase Hypersonic Guidance

The modified Apollo final phase (MAFP) entry guidance algorithm is a bank-to-steer guidance law which is a derivative of the Apollo final phase guidance law used during Earth entry from Lunar return of the Apollo spacecraft.[3] MAFP was developed in response to growing accuracy needs for Martian entry and is baselined to be used for the first time on MSL (scheduled to launch in 2011). In general, bank-to-steer guidance modulates the direction of the lift vector of the entry vehicle around the velocity vector in order to achieve some desired state. For the MAFP algorithm, this desired state is a reference trajectory, comprised of range-to-go, acceleration due to drag, and altitude rate which is stored as a function of the relative velocity. The guidance algorithm is activated once sufficient sensible drag (≥ 0.5 g's) is detected. It then predicts the range-to-go based as a function of the error in the drag and altitude rate, as shown in (69).

$$R_p = R_{ref} + \frac{\partial R}{\partial D}\left(D - D_{ref}\right) - \frac{\partial R}{\partial \dot{r}}\left(\dot{r} - \dot{r}_{ref}\right) \tag{69}$$

The desired vertical component of the lift-to-drag ratio is then able to be calculated as

$$\left(\frac{L}{D}\right)_C = \left(\frac{L}{D}\right)_{ref} + \frac{K_3\left(R - R_p\right)}{\partial R / \partial (L/D)} \tag{70}$$

and finally the bank angle command is given by

$$\sigma_c = \cos^{-1}\left(\frac{(L/D)_C}{L/D}\right) K2ROLL \tag{71}$$

where $K2ROLL$ is a control on the direction of bank angle to account for bank reversals. The gains used in the algorithm are calculated using linear perturbation theory from the reference and are calculated using the adjoint equations of the system. The advantage of the MAFP algorithm is that it has shown to be extremely robust in practice, with a solution existing at every instant in time, which is an advantageous property for flight implementation. Furthermore, it only relies on parameters that are directly sensible (or that can be derived from directly sensible parameters). However, there is no guarantee of optimality on the control effort utilized, which correlates to the propellant mass used to command the bank angle through the trajectory.

VII. State-Dependent Riccati Equation Tracking Controller Development for a High Mass Mars Entry Vehicle

A. System Factorization

The objective of this controller design is to track a full-state reference path, therefore first define the error of the system as

$$\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}} \tag{72}$$

where $\mathbf{x}$ is the current state vector and $\hat{\mathbf{x}}$ is the state vector along the reference trajectory. With this definition, the atmospheric entry system that we wish to control has dynamics governed by

$$\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} \tag{73}$$

The reference path dynamics $\dot{\hat{\mathbf{x}}}$ are determined (and subsequently tabulated) during the generation of the reference trajectory, *apriori* to actual implementing the guidance law and therefor act as a bias to the system, not influencing the system dynamics described in (57)-(64). Because one of the solution procedures pursued will be through a sum-of-squares approach, the extended linearization of the equations needs to contain a polynomial SDC matrix. To achieve this, a Taylor series expansion of the trigonometric and exponential functions is performed such that

$$\sin(x) \approx x - \frac{x^3}{6} + ... \tag{74}$$

$$\cos x \approx 1 - \frac{x^2}{2} + ... \tag{75}$$

$$\tan x \approx 1 + \frac{x^3}{3} + ... \tag{76}$$

$$\exp x \approx 1 + x + ... \tag{77}$$

Let the state vector be defined as

$$\mathbf{x}(t) = [\theta(t)\ \phi(t)\ r(t)\ \psi(t)\ \gamma(t)\ V(t)\ \sigma(t)\ \dot{\sigma}(t)\ m(t))]^T \tag{78}$$

and the control defined as solely the thrust, $T(t)$. Applying the polynomial approximations above, leads to one potential extended linearization for the state SDC matrix of the system as shown in (80) on the subsequent page. The control SDC matrix, for this particular factorization is given by

$$\mathbf{B}(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{d}{I_b} \\ \frac{-1}{I_{sp} g_0} \end{bmatrix} \tag{79}$$

It is important to note that this extended linearization of the flight mechanics of the entry vehicle is not unique, and, in fact, several variations can be seen, by choosing different parameters to factor out. Furthermore, when not wishing to solve the SDRE using a sum-of-squares technique to solve the SDRE was desired, extended linearizations of the system are readily available without the use of any approximations.

B. SDRE Solution Procedure

With the extended linearized system determined, several solution paths can be pursued in order to obtain the closed-loop control solution to guide the vehicle towards the reference trajectory. The first option involves following the following procedure at each point along the trajectory:

1. Evaluate the $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ matrices given in (80) and (79), respectively, at the current value of the state
2. Form the SDRE as shown in (6) where the SDC's are constant matrices and the state and control weighting matrices (*i.e.*, $\mathbf{Q}(\mathbf{x})$ and $\mathbf{R}(\mathbf{x})$) are also constant matrices, determined *apriori*. (For the results shown in this report, they are arbitrarily taken to be identity)
3. Solve the SDRE evaluated at the current state as if it were a linear Riccati equation, for the positive-definite matrix, $\mathbf{P}(\mathbf{x})$
4. Invoke the state-feedback control law given in (7) as the control

Note that provided a solution for $\mathbf{P}(\mathbf{x})$ exists, this solution method assures local asymptotic stability to the desired reference trajectory. However, there is no guarantee that the extended linearization chosen will always result in an obtainable solution for the desired reference trajectory (and hence at some points in the trajectory a control will not be able to be found). This method solves the semidefinite programming problem for the state-feedback control at each point along the trajectory. Because this technique involves using semidefinite programming, an external program, SOSTOOLS v2.01[13] coupled with SeDuMi,[14] was utilized. This tool combination allows for direct synthesis of the sum-of-squares formulation of the SDRE, by solving the LQR problem using Shur decomposition real-time with the trajectory simulation. The outline of the solution procedure is as follows:

1. Formulate the appropriate polynomial objects in SOSTOOLS for the extended linearization representation of the dynamics
2. Choose appropriate values for the $\mathbf{C_1}(\mathbf{x})$ and $\mathbf{C_2}(\mathbf{x})$ matrix (For the results shown in this report, these are taken to be identity)
3. Call SOSTOOLS from the trajectory simulation to evaluate whether or not a solution satisfying Theorem 6
4. If a solution exists, evaluate the feedback-control law polynomial object returned (valid for a local region of the current condition), if a solution does not exist exit with no obtainable solution

Note that the solution obtained through the sum-of-squares approach is a local solution through the options in SOSTOOLS with updates being recomputed every 30 seconds along the trajectory (approximately 10 times throughout the entry) to ensure that the locally obtained solution is valid.

$$\mathbf{A}(\mathbf{x}) = [\mathbf{A_{1-5}}(\mathbf{x}) \quad \mathbf{A_{6-9}}(\mathbf{x})] \tag{80}$$

$$\mathbf{A_{1-5}}(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 & 0 & \left(\frac{\gamma^2\psi^2}{4} - \frac{\gamma^2}{2} - \frac{\psi^2}{2} + 1\right)\frac{1}{r-\frac{\phi^2 r}{2}} \\ 0 & 0 & 0 & \frac{V}{r} - \frac{V\psi^2}{6r} + \frac{V\gamma^2\psi^2}{12r} - \frac{V\gamma^2}{2r} & 0 \\ 0 & 0 & 0 & 0 & \frac{V\gamma^2}{6} - V \\ 0 & -\frac{V\phi^2}{3r} - \frac{V\gamma^2\phi^2}{6r} + \frac{V\phi^2\psi^2}{6r} - \frac{V\gamma^2\phi^2\psi^2}{12r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{g_p r_p^2}{r^2} - \frac{\gamma^2 g_p r_p^2}{6r^2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{g_p r_p^2}{r^2} - \frac{\gamma^2 g_p r_p^2}{6r^2} \\ \frac{1}{\theta} & 0 & 0 & 0 & 0 \\ \frac{1}{\theta} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{81}$$

$$\mathbf{A_{6-9}}(\mathbf{x}) = \begin{bmatrix} \left(\frac{\gamma^2\psi^2}{4} - \frac{\gamma^2}{2} - \frac{\psi^2}{2} + 1\right)\frac{1}{r-\frac{\phi^2 r}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{C_L V S \rho_p r_p}{mr}\left(\frac{r_p}{2} - \frac{V\gamma^2}{4} - \frac{1}{12} + \frac{\gamma^2\sigma^2}{24}\right) & 0 & 0 & 0 \\ 0 & \frac{V}{2r} - \frac{1}{r} + \frac{\psi^2}{2r} - \frac{\gamma^2\psi^2}{4r} & 0 & -\frac{C_D S V^2 \rho_p r_p}{2m^2 r} \\ -\frac{\gamma g_p r_p^2}{2Vr^2} & \frac{\gamma^2}{2r} - \frac{1}{r} + \frac{g_p r_p^2}{V^2 r^2} - \frac{C_L S \rho_p r_p}{2mr} + \frac{C_L S \rho_p r_p \sigma^2}{4mr} & 0 & 0 & 0 \\ 0 & -\frac{C_D S V \rho_p r_p}{2mr} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{82}$$

VIII. High Mass Mars Entry Vehicle Hypersonic Guidance Performance

A. Reference Trajectory

A single reference trajectory was designed in order to evaluate the performance of each algorithm evaluated. The SDRE based guidance laws track the full state vector along the reference, while MAFP follows four states (or derived states)–range-to-go, acceleration due to drag, altitude rate, and velocity. The reference trajectory is defined at a constant $45°$ bank angle from the initial conditions seen in Table 2 and is terminated at the final conditions seen in Table 3 (nominally). In altitude-velocity space, the reference trajectory is included as part of the plot seen in Figure 1.

Table 2. Reference Trajectory Initial Conditions

Parameter	Description	Value	Units
θ_0	Initial Longitude	0	deg
ϕ_0	Initial Latitude	0	deg
h_0	Initial Altitude	125	km
ψ_0	Initial Azimuth Angle	90	deg
γ_0	Initial Flight Path Angle	10	deg
V_0	Initial Velocity	4500	m/s
m_0	Initial Mass	2196	kg
σ_0	Initial Bank Angle	45	deg

Table 3. Reference Trajectory Final Conditions

Parameter	Description	Value	Units
θ_f	Final Longitude	-1.3947	deg
ϕ_f	Final Latitude	16.279	deg
h_f	Final Altitude	7	km
ψ_f	Final Azimuth Angle	113.7	deg
γ_f	Final Flight Path Angle	28.8	deg
V_f	Final Velocity	492.4	m/s
m_f	Final Mass	2196	kg
σ_f	Final Bank Angle	45	deg

B. Region of Attraction

Using the adjoint method outlined previously (*i.e.*, invoking Proposition 1), the region of attraction around the reference trajectory was able to be estimated. The solution procedure involves integrating the Hamiltonian dynamics as well as the adjoint equations of the system from the final state (taken to be the full state defined at 7 km, which is approximately the planned parachute deployment condition for MSL) while attempting to find the largest region in state space which satisfies (22). The region of attraction was numerically approximated for each state variable by using a Newton-Raphson iteration on (22) with a convergence criterion of 1% of the state values along the reference trajectory. The results of this estimation when taken to altitude-velocity space are shown in Figure 1. It is interesting to note that for the reference trajectory used in this analysis, that virtually no region attraction exists above 83 km. This is somewhat expected as the control authority is negligible in this regime with little dynamic pressure to yield significant lift.

C. Continuous and Piecewise Continuous SDRE Performance

In order to gain a sense of the performance of SDRE control for hypersonic aeromaneuving of entry vehicles on Mars, as well as the performance of the sum-of-squares decomposition relative to solving the SDRE with the SDC

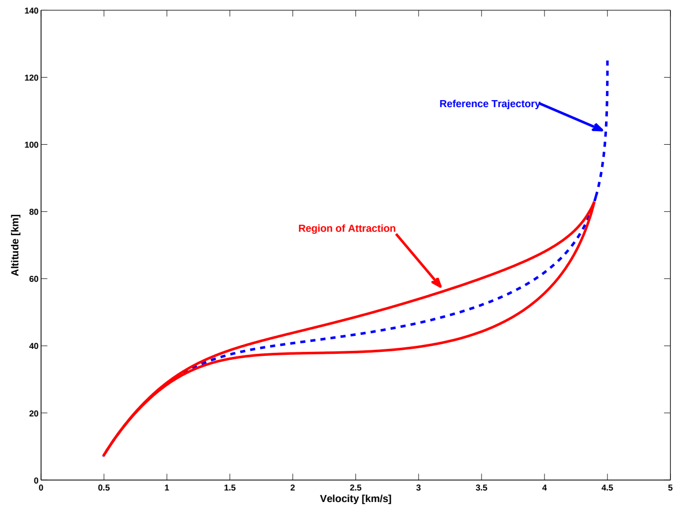

Figure 1. Region of Attraction Around the Reference Trajectory

matrices updated continually with the current state, the two different implementations of the SDRE were implemented. Additionally, MAFP was included to serve as a performance baseline relative to the current state of the art for flight implementable hypersonic guidance. Two different scenarios were analyzed, one in which the density was 10% higher than that modeled in the nominal (reference) trajectory and one in which the entry flight path angle is shallowed to $8°$. Performance data is shown in Table 4 and 5 for the two different cases. While Figure 2 and Figure 3 shows the trajectory for each of the control laws in altitude-velocity space for the increased density case and shallow flight path angle case, respectively. Also shown is how the error vector, $\mathbf{e}$, evolves with velocity for each of the algorithms. This error vector is normalized by the maximum magnitude observed for each algorithm, to make the results more relevant. This velocity evolution is shown in Figure 4 and Figure 5. In the results, modified Apollo final phase is identified as *MAFP*, the SDRE with SDC matrices continuously updated is identified as *Continuous*, and the SDRE solved locally using the sum-of-squares technique is identified as *Sum-of-squares*.

Table 4. Performance Comparison with 10% Higher Density

Algorithm	**Velocity Error at 7 km (m/s)**	**Position Error at 7 km (m)**	**Propellant Usage (kg)**
MAFP	8.6	398	33.9
Continuous	7.5	420	27.6
Sum-of-squares	9.4	440	29.2

Table 5. Performance Comparison with $\gamma = 8°$

Algorithm	**Velocity Error at 7 km (m/s)**	**Position Error at 7 km (m)**	**Propellant Usage (kg)**
MAFP	16.3	287	28.9
Continuous	9.6	378	27.6
Sum-of-squares	29.6	622	31.2

It is seen that all three algorithms had similar performance with the sum-of-squares algorithm performing poorer than the continuous SDRE algorithm for both cases examined. This is can be attributed to the update rate of the local solution. Furthermore, it is seen that all three algorithms perform similarly in how the negate the relative error, with

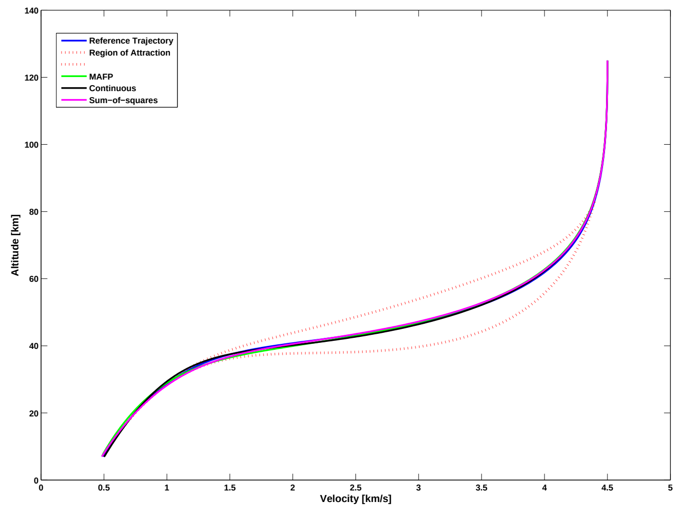

Figure 2. Trajectory Comparison with 10% Higher Density

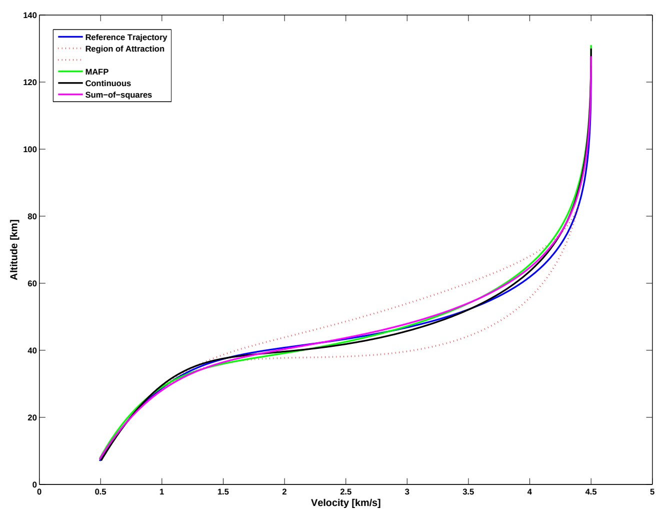

Figure 3. Trajectory Comparison with $\gamma = 8^\circ$

early error growth being primarily dominated by the timeline (range) differing dramatically from that of the reference.

IX. Conclusions and Future Work

Through this work, a SDRE control solution to the bank-to-steer hypersonic aeromaneuvering of a high mass entry vehicle at Mars was accomplished. An analysis of the solution tools available was conducted as well as the performance relative to a flight-heritage bank-to-steer guidance algorithm. While the guidance accuracy for all algorithms was acceptable, a propellant benefit was seen to implementing the SDRE as it solves a sub-optimal LQR problem that accounts for control effort directly. However, this is at the cost of having the reference the full state of the system as opposed to a minimal state, that is fully sensible in flight. A sum-of-squares technique which locally solved a generic control problem was observed to give slightly poorer performance relative to that of a continuous linear LQR prob-

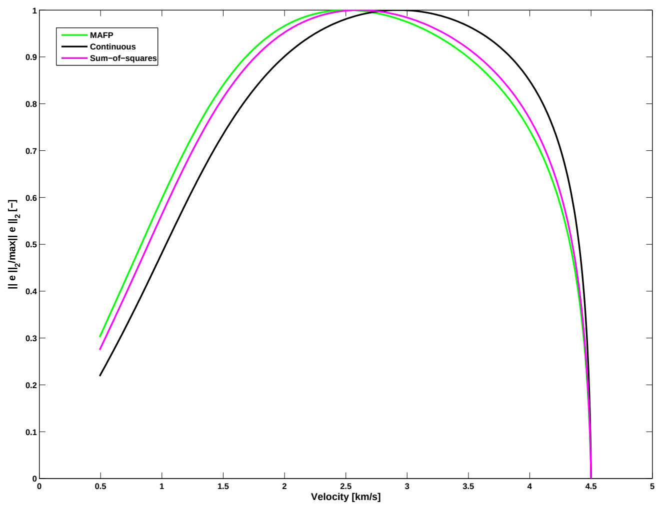

Figure 4. Error Vector Magnitude Comparison with 10% Higher Density

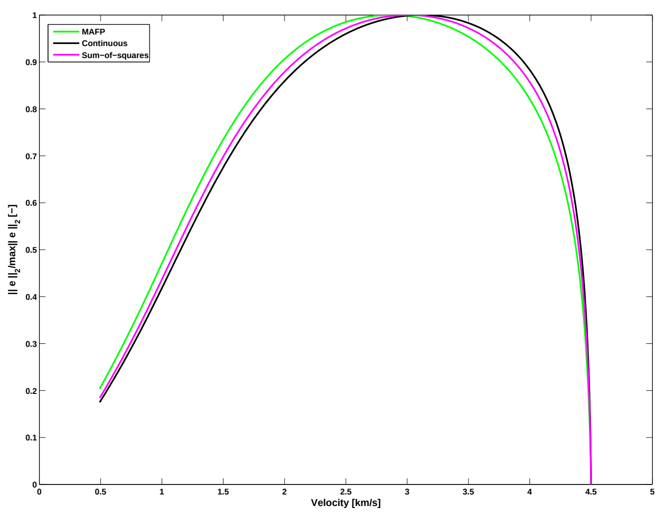

Figure 5. Error Vecotr Magnitude Comparison with $\gamma = 8°$

lem. In the future, an investigation into alternative extended linearizations of the system as well as the impact of the Taylor series polynomial approximation should be performed in order to fully characterize the quality of the solution obtained.

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