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INVARIANT THEORY AS A TOOL FOR SPACECRAFT NAVIGATION

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Many spacecraft navigation algorithms are built upon models describing the geometric relationships between the spacecraft state and the measurement produced by a sensor. This is especially true for vision-based sensors. However, extracting the maximum amount of independent state information from a measurement (or set of measurements) is not always straightforward. This work investigates the utility of invariant theory as a tool to better utilize the information content within sensor data for spacecraft navigation. Direct applications include star pattern recognition, terrain relative navigation (TRN), and LIDAR point cloud registration.

INTRODUCTION

A major part of spacecraft navigation and orbit determination (OD) is using sensor data to infer the state of a spacecraft. In some cases, such as with Inertial Measurement Units (IMUs), this sensor data may be collected and processed with only limited consideration of the outside world. In most cases, however, the navigation sensors (e.g., cameras, LIDARs, RF transceivers) observe a phenomena whose value is functionally dependent on the relative geometry between the spacecraft and other objects. It is exactly this functional dependence—the fact that a change in sensor reading informs us of a change in the spacecraft state—that makes these observations useful for state estimation.

However, these changing measurement values sometimes make data association difficult when attempting to establish correspondence between two data sets. For example, given a set of star observations in an image, how do we establish correspondence (i.e., match) each observed star to a catalog of known stars? Moreover, how do we know if any of the observations arise from objects (stars or otherwise) not existing within our catalog (i.e., are any of the observations outliers)? These questions describe the so-called "lost-in-space star identification" or "astrometric calibration" problem that has spawned 100s of proposed asterism matching schemes over the past 50 years [1, 2, 3, 4]. Similar measurement-to-catalog matching problems exist for crater identification [5], natural (arbitrary) landmark identification [6], spacecraft retroreflector identification [7], and a variety of other spaceflight applications. As another example, we might collect a sequence of measurements and wish to know if these are observations of the same object or of different objects—a problem often arising in orbit determination [8]. Regardless of the application, correct correspondence is critical since most state estimation frameworks used in practice (e.g., least squares, Kalman filters) do not provide a way of accounting for this kind of misrepresentation in the measurement model. While some recent techniques may help soften this dependence in certain situations [9], correct measurement correspondence (either explicit or implicit) is ultimately necessary.

Many of the feature descriptors schemes in use today for visual recognition of objects are heuristic, ad hoc, and/or brute force—but this need not be the case. They key to developing a rigorous framework is to first recognize that much of what we seek in practical navigation is geometric in nature and then to exploit the deep relationship between this geometry and the concept of invariants. Indeed, this relationship is described

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by the Klein-Weyl Thesis [10]: "The only important geometric relationships are defined by invariants and conversely every invariant has a significant and interesting role in geometric analysis."

Therefore, this work explores the fundamental role invariant theory plays in the geometry of spacecraft navigation. Our approach utilizes concepts from algebraic geometry to consider functions of a particular algebraic variety (e.g., points, planes, conics) whose result does not change (i.e., is *invariant*) under a particular group action [e.g., $\mathrm{SO}(3)$, $\mathrm{SE}(3)$, $\mathrm{PGL}(3)$]. Many popular navigation algorithms—especially in the areas of star identification, terrain relative navigation (TRN), and relative navigation (RelNav)—rely on results from invariant theory, even if this dependence is implicit and not often recognized. When we fail to deeply understand a problem's geometry, the result is an algorithm that is incomplete, has a narrow applicability, or is a duplication of an existing algorithm in disguise. This work illustrates the utility of invariant theory for avoiding some of these common pitfalls. We also provide a survey of results (some known, some novel) for a wide variety of generic problems, and hope this becomes a useful guide for future algorithm developers.

GEOMETRY AND INVARIANTS

In spacecraft navigation we seek to extract information from sensor measurements of the surrounding environment to infer the vehicle's state. Suppose, therefore, that we describe an object in the surrounding environment by the model $\mathcal{M}$. If we aim to recognize this object when observed from an arbitrary vantage point, we are especially interested in attributes that are independent of the position and orientation (i.e., pose) of the sensor. That is, we are interested in functions that depend on the model that remain unchanged when the sensor moves relative to the model. More than that, we also seek functions describing attributes that persist across the measurement generation process (e.g., attributes that may be constructed from the model or from images of the model).

Equivalence Relations and Invariants on Models

Without loss of generality, we can take an observer-centric viewpoint, where we fix the sensor and move the model relative to the sensor. Thus, the motion of the sensor relative to the world may be described by action of the group G on the set of models. A group element $g \in G$ transforms a model $\mathcal{M}$ to another model $g \cdot \mathcal{M}$. In the case of spacecraft navigation (and in many terrestrial robotic applications), the relative motion between the sensor and the model is governed by the group $G = \mathrm{SE}(3)$ of isometries of $\mathbb{R}^3$ consisting of rotations and translations (i.e., rigid relative motion). Thus, since we seek model attributes that do not change under the action of group G, our interest lies in functions that satisfy $f(g \cdot \mathcal{M}) = f(\mathcal{M})$ for every $g \in G$ and every model $\mathcal{M}$. Such a function is called an *invariant* (or G-invariant).

From an object recognition, we wish to make no distinction between a model $\mathcal{M}$ and a transformed (e.g., translated, rotated) version of this same model given by $\mathcal{M}' = g \cdot \mathcal{M}$. Thus, we call two models $\mathcal{M}$ and $\mathcal{M}'$ equivalent (notation: $\mathcal{M} \sim \mathcal{M}'$) if there exists a group element $g \in G$ with $\mathcal{M}' = g \cdot \mathcal{M}$. Invariants $f_1, f_2, \ldots, f_s$ form a *complete system of invariants* if $\mathcal{M} \sim \mathcal{M}'$ if and only if $f_i(\mathcal{M}) = f_i(\mathcal{M}')$ for $i = 1, 2, \ldots, s$. If $f_1, f_2, \ldots, f_s$ is a complete system of invariants, then to test whether $\mathcal{M} \sim \mathcal{M}'$ we only have to evaluate $f_1, f_2, \ldots, f_s$ on $\mathcal{M}$ and $\mathcal{M}'$. The fact that the values of the invariants determine the model up to equivalence is in harmony with the Klein-Weyl Thesis mentioned in the introduction.

The relation $\sim$ is an equivalence relation, which means that:

1. $\sim$ is reflexive: $\mathcal{M} \sim \mathcal{M}$ (because $\mathcal{M} = 1 \cdot \mathcal{M}$ where $1 \in G$ is the identity);
2. $\sim$ is symmetric; if $\mathcal{M} \sim \mathcal{M}'$, then $\mathcal{M}' \sim \mathcal{M}$ (because if $\mathcal{M}' = g \cdot \mathcal{M}$, then $\mathcal{M} = g^{-1} \cdot \mathcal{M}'$, where g^{-1} is the inverse of g);
3. $\sim$ is transitive: if $\mathcal{M} \sim \mathcal{M}'$ and $\mathcal{M}' \sim \mathcal{M}''$, then $\mathcal{M} \sim \mathcal{M}''$ because if $\mathcal{M}' = h \cdot \mathcal{M}$ and $\mathcal{M}'' = g \cdot \mathcal{M}'$, then $\mathcal{M}'' = gh \cdot \mathcal{M}$).

The equivalence class $[\mathcal{M}]$ of some model $\mathcal{M}$ is the set of all models $\mathcal{M}'$ with $\mathcal{M} \sim \mathcal{M}'$. From the definition of $\sim$ is clear that $[\mathcal{M}]$ is exactly the orbit $\{g \cdot \mathcal{M} \mid g \in G\}$ of $\mathcal{M}$.

As an example, we consider models of the form $\mathcal{M} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_d)$, where $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_d$ are points in $\mathbb{R}^3$. This model arises often in practice, such as a 3D object parameterized by the location of 3D keypoints residing on its surface. Sometimes these keypoints are artifical (e.g., retroreflectors on the ISS [7, 11]) and sometimes they are naturally occurring interest points (e.g., maplets on astroid surfaces [6] or 3D features on LIDAR data [12, 13]). Other models—such as lines, conics, and planes—are also important, but we defer their discussion to later sections. If we allow the model $\mathcal{M}$ to translate, rotate, and/or reflect according to the symmetry group $G = \mathrm{E}(3)$, then the pairwise distances $\delta_{i,j}(\mathcal{M}) = \|\boldsymbol{p}_i - \boldsymbol{p}_j\|$ are invariants. To make this explicit, we parameterize $\mathrm{E}(3)$ by the 3×3 orthogonal matrix $\boldsymbol{A} \in \mathrm{O}(3)$ and the translation $\boldsymbol{r} \in \mathbb{R}^3$ such that the sensor-relative location of model point $\boldsymbol{p}_i$ is given by $\boldsymbol{q}_i$,

$$\boldsymbol{q}_i = \boldsymbol{A}\boldsymbol{p}_i + \boldsymbol{r} \tag{1}$$

The transformed model is then $\mathcal{M}' = (\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_d)$. It is straightforward to show that the pairwise distances are invariant to the action of $G = \mathrm{E}(3)$,

$$\delta_{i,j}(\mathcal{M}') = \|\boldsymbol{q}_i - \boldsymbol{q}_j\| = \|(\boldsymbol{A}\boldsymbol{p}_i + \boldsymbol{r}) - (\boldsymbol{A}\boldsymbol{p}_j + \boldsymbol{r})\| = \|\boldsymbol{A}(\boldsymbol{p}_i - \boldsymbol{p}_j)\| = \|\boldsymbol{p}_i - \boldsymbol{p}_j\| = \delta_{i,j}(\mathcal{M}) \tag{2}$$

Thus, for a d-tuple of 3D points under the symmetry of $\mathrm{E}(3)$, the invariants $\{\delta_{i,j} \mid 1 \leq i < j \leq d\}$ form a complete system of invariants, because the model $\mathcal{M}$ can be reconstructed up to isometry from the pairwise distances $\delta_{i,j}(\mathcal{M})$. We note that the pairwise distances do not form a complete system of invaraints for rigid transformations $G = \mathrm{SE}(3)$ since pairwise distances alone result in a reflection ambiguity not present in $\mathrm{SE}(3)$. This reflection ambiguity may not matter in some applications if reflections are unimportant or rarely encountered in practice.

Equivalence Relations, Invariants, and View Invariants on Measurements

Navigation systems work by using information collected by sensors that observe the model $\mathcal{M}$. Thus, since all we have is sensor measurements, practically useful invariants must survive the measurement generation process and be reproducible from only the measured data. This is step is often trivial for 3D sensors (e.g., LIDARs) that directly measure the sensor-relative 3D location of model points [i.e., directly measures $\boldsymbol{q}_i$ from Eq. (1)]. More challenging is the situation when all the measurements are from cameras.

In vision-based spacecraft navigation, we typically have an image (or set of images) $\mathcal{I}$ that is some kind of projection image $\pi(\mathcal{M})$ of a model $\mathcal{M}$. For example, π could be a stereographic, orthogonal, or perspective projection. A central question is: given two images $\mathcal{I}$ and $\mathcal{I}'$, are they projections of two models $\mathcal{M}$ and $\mathcal{M}'$ that are equivalent? We can define a relation $\equiv$ on the set of images, where $\mathcal{I} \equiv \mathcal{I}'$ if and only if there exist models $\mathcal{M}$ and $\mathcal{M}'$ with $\mathcal{I} = \pi(\mathcal{M})$, $\mathcal{I}' = \pi(\mathcal{M}')$, and $\mathcal{M} \sim \mathcal{M}'$.

In some cases a model can be reconstructed from the image $\mathcal{I} = \pi(\mathcal{M})$, especially in the case where $\mathcal{I}$ consists of more than one image. If this is the case, then one can verify whether $\mathcal{I} \equiv \mathcal{I}'$ by first reconstructing the models $\mathcal{M}$ and $\mathcal{M}'$ and then verifying whether $\mathcal{M} \sim \mathcal{M}'$ using a complete system of invariants. There are many examples where models can be constructed from one or more images. In the case where models can be reconstructed from images, $\equiv$ is an equivalence relation as well.

If we cannot reconstruct the model from the image, then the relation $\equiv$ may not be an equivalence relation because it may not be transitive (but it will still be reflexive and symmetric). As an example, consider again the models consisting of point clouds in $\mathbb{R}^3$ with d points, and consider images that are orthogonal projections. Suppose $\mathcal{I} = (\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_d)$ and $\mathcal{I}' = (\boldsymbol{u}'_1, \boldsymbol{u}'_2, \ldots, \boldsymbol{u}'_d)$ are two images, where $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d, \boldsymbol{u}'_1, \ldots, \boldsymbol{u}'_d$ are the measured image points in $\mathbb{R}^2$. If $\mathcal{I} \equiv \mathcal{I}'$ then there exists points $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_d \in \mathbb{R}^3$, 2×3 matrices $\boldsymbol{A}$, $\boldsymbol{A}'$ and vectors $\boldsymbol{b}, \boldsymbol{b}' \in \mathbb{R}^2$ such that $\boldsymbol{u}_i = \boldsymbol{A}\boldsymbol{p}_i + \boldsymbol{b}$ and $\boldsymbol{u}'_i = \boldsymbol{A}'\boldsymbol{p}_i + \boldsymbol{b}'$ for all i, so the rank of

$$\begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_d \\ \boldsymbol{u}'_1 & \boldsymbol{u}'_2 & \cdots & \boldsymbol{u}'_d \\ 1 & 1 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{A}' & \boldsymbol{b}' \\ \boldsymbol{0}_{3\times 1} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & \cdots & \boldsymbol{p}_d \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

is at most 4, because the first matrix on the right-hand side has only 4 columns. For this example and for $d = 5$, we can define a function r by

$$r(\mathcal{I}, \mathcal{I}') = \det \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_5 \\ \boldsymbol{u}_1' & \boldsymbol{u}_2' & \cdots & \boldsymbol{u}_5' \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Then $r(\mathcal{I}, \mathcal{I}') \neq 0$ implies that $\mathcal{I} \not\equiv \mathcal{I}'$. We call r a relational function. For example, the reader may verify the $r(\mathcal{I}, \mathcal{I}') \neq 0$, when

$$\mathcal{I} = \left(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\right), \qquad \mathcal{I}' = \left(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\right). \tag{3}$$

Ideally, we have a complete system of relational functions $r_1, r_2, \ldots, r_t$ in the sense that $\mathcal{I} \equiv \mathcal{I}'$ if and only if $r_k(\mathcal{I}, \mathcal{I}') = 0$ for $k = 1, 2, \ldots, t$. Although a system $r_1, r_2, \ldots, r_t$ is useful to compare two images, it not as useful for matching an image $\mathcal{I}'$ to images within a large catalog $\{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_N\}$. One could evaluate $r_j(\mathcal{I}', \mathcal{I}_k)$ for $j = 1, 2, \ldots, t$ and $k = 1, 2, \ldots, N$ but this could be too time consuming when N is large.

Invariants are more powerful than relations for object recognition. Suppose for example that models can be reconstructed from the images. So then we could reconstruct models $\mathcal{M}', \mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_N$ (up to the action of the symmetry group G) from the images $\mathcal{I}', \mathcal{I}_1', \mathcal{I}_2', \ldots, \mathcal{I}_N'$. Now $\mathcal{M}' \sim \mathcal{M}_k$ if and only if $\mathcal{I}' \equiv \mathcal{I}_k$. Suppose that $f_1, f_2, \ldots, f_s$ is a complete system of invariants, and let $f = (f_1, f_2, \ldots, f_s)$. To match an image $\mathcal{I}'$ (and corresponding model $\mathcal{M}'$) to an image in the database, we have to find a k such that $f(\mathcal{M}') = f(\mathcal{M}_k)$ for some k. The values of the invariants $f(\mathcal{M}_1), f(\mathcal{M}_2), \ldots, f(\mathcal{M}_N)$ can be pre-computed and stored in a database. Given an image $\mathcal{I}'$, one can reconstruct a model $\mathcal{M}'$ with $\pi(\mathcal{M}') = \mathcal{I}'$, compute $f(\mathcal{M}')$ and quickly match it in the database. A quick match is made possible (e.g., with a nearest neighbor search) since $f(\mathcal{M}') = f(\mathcal{M}_k)$ and $f(\mathcal{M}') \neq f(\mathcal{M}_i)$ for $i \neq k$. We find, however, that explicit reconstruction of the model $\mathcal{M}$ is not always convenient or possible—and so we search for an even more powerful technique for image-to-catalog matching.

View invariants (when they exist) are even more powerful than invariants since they permit direct image-to-image, image-to-model, and/or image-to-catalog matching, without the need for model reconstruction. This idea is now developed. If $\equiv$ is not an equivalence relation, then we can still construct a different equivalence relation $\equiv\!\equiv$ from $\equiv$ by using transitive closure. We define $\mathcal{I} \equiv\!\equiv \mathcal{I}'$ if and only if there is a finite sequence of images $\mathcal{I}_0, \mathcal{I}_1, \cdots, \mathcal{I}_k$ such that $\mathcal{I} = \mathcal{I}_0 \equiv \mathcal{I}_1 \equiv \cdots \equiv \mathcal{I}_k = \mathcal{I}'$. We may not be able to use all invariants, because the invariants have to be evaluated on models and the models cannot be reconstructed. Some invariants, which we will call view invariants, can be computed from the an image. A function h on images is a *view invariant* if there exist an invariant f of models such that $f(\mathcal{M}) = h(\mathcal{I})$ whenever $\mathcal{I} = \pi(\mathcal{M})$ is an image of the model $\mathcal{M}$. If $\mathcal{I} \equiv \mathcal{I}'$, say $\mathcal{I} = \pi(\mathcal{M})$, $\mathcal{I}' = \pi(\mathcal{M}')$ and $\mathcal{M} \sim \mathcal{M}'$, then $h(\mathcal{I}) = f(\mathcal{M}) = f(\mathcal{M}') = h(\mathcal{I}')$. If $\mathcal{I} \equiv\!\equiv \mathcal{I}$, say say $\mathcal{I} = \mathcal{I}_0 \equiv \mathcal{I}_1 \equiv \cdots \equiv \mathcal{I}_k = \mathcal{I}'$, then we have $h(\mathcal{I}) = h(\mathcal{I}_0) = h(\mathcal{I}_1) = \cdots = h(\mathcal{I}_k) = h(\mathcal{I}')$. Such view invariants are very useful because we can compute them from an image and they allow us to look up a model within a large database. =A quick match is made possible (e.g., with a nearest neighbor search) since $h(\mathcal{I}') = f(\mathcal{M}_k)$ and $h(\mathcal{I}') \neq f(\mathcal{M}_i)$ for $i \neq k$. Moreover, view invariants are useful for determining if two images $\mathcal{I}$ and $\mathcal{I}'$ are observations of the same object by simply checking if $h(\mathcal{I}) = h(\mathcal{I}')$, which may be accomplished even if the specific underlying model $\mathcal{M}_k$ is never known (and, possibly, cannot be reconstructed). Unfortunately, despite their utility, such view invariants may not always exist.

In the example of d-point clouds in $\mathbb{R}^3$ with symmetry group $\mathrm{E}(3)$ and images that are orthogonal projections, there are no view invariants. To see this, Suppose that

$$\mathcal{I} = \left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}, \ldots \begin{bmatrix} x_d \\ y_d \end{bmatrix}\right), \qquad \mathcal{I}' = \left(\begin{bmatrix} z_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ w_2 \end{bmatrix}, \ldots, \begin{bmatrix} z_d \\ w_d \end{bmatrix}\right)$$

are two arbitrary images of d points. Let

$$\mathcal{I}'' = \left(\begin{bmatrix} y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} y_2 \\ z_2 \end{bmatrix}, \ldots \begin{bmatrix} y_d \\ z_d \end{bmatrix}\right), \mathcal{M} = \left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, \ldots, \begin{bmatrix} x_d \\ y_d \\ z_d \end{bmatrix}\right), \mathcal{M}' = \left(\begin{bmatrix} y_1 \\ z_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} y_2 \\ z_2 \\ w_2 \end{bmatrix}, \ldots, \begin{bmatrix} y_d \\ z_d \\ w_d \end{bmatrix}\right).$$

If $\pi_{1,2} : \mathbb{R}^3 \to \mathbb{R}^2$ and $\pi_{2,3} : \mathbb{R}^3 \to \mathbb{R}^2$ are the projections onto the first 2 coordinates, and the last 2 coordinates respectively, then we have $\pi_{1,2}(\mathcal{M}) = \mathcal{I}$ and $\pi_{2,3}(\mathcal{M}) = \mathcal{I}''$, so $\mathcal{I} \equiv \mathcal{I}''$. Moreover, $\pi_{1,2}(\mathcal{M}') = \mathcal{I}''$ and $\pi_{2,3}(\mathcal{M}') = \mathcal{I}'$, so $\mathcal{I}'' \equiv \mathcal{I}$. It follows that $\mathcal{I} \equiv\equiv \mathcal{I}'$. This means that for every view invariant h, we have $h(\mathcal{I}) = h(\mathcal{I}')$. In other words, there are only trivial view invariants. In the example (3) earlier, we had two images $\mathcal{I}, \mathcal{I}'$ with $\mathcal{I} \not\equiv \mathcal{I}'$ for $d = 5$. Since $\mathcal{I} \equiv\equiv \mathcal{I}'$ we see that $\equiv$ is not an equivalence relation.

Another interesting example is where the models are d points in $\mathbb{R}^3$ that are coplanar, and the projection π is perspective projection. A model $\mathcal{M}$ cannot be reconstructed from an image $\mathcal{I} = \pi(\mathcal{M})$ up to the group $\mathrm{E}(3)$ of Euclidean symmetries. However, there are view invariants in this case. We embed $\mathbb{R}^3$ into 3-dimensional projective space $\mathbb{P}^3$ on which the group $\mathrm{PGL}(4)$ acts. Although the model $\mathcal{M}$ cannot be reconstructed up to $\mathrm{E}(3)$-symmetry, it can be reconstructed up to $\mathrm{PGL}(4)$ symmetry. In this case, we obtain view invariants from invariants of models for the larger symmetry group $\mathrm{PGL}(4)$.

DIFFERENT TYPES OF INVARIANTS

There are an assortment of different types of invariants that one might construct, depending on the functional form of the invariant function f. Historically, much of the focus of invariant theory has been on polynomial invariants. We quickly discover, however, that such polynomial invariants are not appropriate for all applications (including many computer vision and spacecraft navigation applications), and so we consider rational invariants. Finally, we generalize further to consider the case of algebraic invariants.

Polynomial Invariants

To develop a complete understanding, we will begin by considering polynomial invariants. Typically one starts with a group G acting on a vector space $\mathbb{R}^d$. We write $\mathbb{R}[x_1, x_2, \ldots, x_d]$ for the ring of polynomials in n variables with real coefficients. A polynomial $f \in \mathbb{R}[x_1, x_2, \ldots, x_d]$ is invariant under G if $f(\mathcal{M}) = f(g \cdot \mathcal{M}')$ for every vector $\mathcal{M} \in \mathbb{R}^d$ and every $g \in G$. The set of all invariant polynomials form a subring of $\mathbb{R}[x_1, x_2, \ldots, x_d]$ (called the *invariant ring*) that is denoted by $\mathbb{R}[x_1, x_2, \ldots, x_d]^G$. If two points $\mathcal{M}$ and $\mathcal{M}'$ are in the same G-orbit, then $f(\mathcal{M}) = f(\mathcal{M}')$ for every polynomial invariant. The converse is not always true in general, but it is true when G is a compact group [14]. One goal in invariant theory is to find a system of *fundamental polynomial invariants*, which is a set of polynomial invariants $f_1, f_2, \ldots, f_r$ such that every polynomial invariant f can be expressed in the form $h(f_1, f_2, \ldots, f_r)$ where h is a polynomial in r variables. We also say that $f_1, f_2, \ldots, f_r$ are generators of the invariant ring $\mathbb{R}[x_1, x_2, \ldots, x_d]^G$. If $f_1, f_2, \ldots, f_r$ is a system of fundamental polynomial invariants, and $f_i(\mathcal{M}) = f_i(\mathcal{M}')$ for $i = 1, 2, \ldots, r$ then it is clear that $f(\mathcal{M}) = f(\mathcal{M}')$ for all polynomial invariants. David Hilbert proved in 1890 [15] that there always exists a finite system of fundamental polynomial invariants if G is compact. However, Nagata [16] showed that there may not be such a finite system if G is not compact, which also gave a counterexample to the 14th problem on Hilbert's famous list of challenges for the 20th century. In the situations that arise in computer vision, there usually is a finite system of fundamental polynomial invariants even in cases where the symmetry group is not compact.

To illustrate this approach in invariant theory we return to our earlier example of the action of the Euclidean group $\mathrm{E}(3)$ on models $\mathcal{M} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_d)$, where

$$\boldsymbol{p}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \in \mathbb{R}^3, \quad i = 1, 2, \ldots, d.$$

We saw in Eq. (2) that some invariants are given by

$$\delta_{i,j}(\mathcal{M}) = \|\boldsymbol{p}_i - \boldsymbol{p}_j\| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}, \quad 1 \le i < j \le d.$$

Now $\delta_{i,j}$ is not a polynomial function because of the square root, but it is still an algebraic function. If we take the squares we get polynomial invariants $\Delta_{i,j} := \delta_{i,j}^2$, $1 \le i < j \le d$. The set of all polynomial functions in the coordinates with real coefficients form the polynomial ring $\mathbb{R}[x_1, y_1, z_1, \ldots, x_d, y_d, z_d]$. For this example, the polynomial invariants $\Delta_{i,j}$ generate the invariant ring, i.e.,

$$\mathbb{R}[\Delta_{1,2}, \Delta_{1,3}, \ldots, \Delta_{1,d}, \Delta_{2,3}, \ldots, \Delta_{d-1,d}] = \mathbb{R}[x_1, y_1, z_1, \ldots, x_d, y_d, z_d]^{\mathrm{E}(3)}.$$

We have $\binom{d}{2}$ fundamental polynomial invariants. For $d \geq 8$ the number of fundamental polynomial invariants is greater than $3d$, the number of parameters in the model. This suggests that some of the fundamental polynomial invariants we have found are superfluous. However, if we remove one of these generators $\Delta_{i,j}$, $1 \leq i < j \leq d$, then they will no longer form a system of fundamental polynomial invariants. In other words, none of the fundamental polynomial invariants can be expressed as a polynomial in the other fundamental polynomial invariants. Nevertheless, it is still possible that one of the fundamental polynomial invariants is determined or almost determined by the other fundamental polynomial invariants because there are algebraic relations among the invariants. For example, if $d = 5$ then there is an algebraic relations among the $\Delta_{i,j}$. If A is the 3×4 matrix

$$\begin{bmatrix} \boldsymbol{p}_1 - \boldsymbol{p}_5 & \boldsymbol{p}_2 - \boldsymbol{p}_5 & \boldsymbol{p}_3 - \boldsymbol{p}_5 & \boldsymbol{p}_4 - \boldsymbol{p}_5 \end{bmatrix},$$

then

$$2A^tA = 2\left[(\boldsymbol{p}_i - \boldsymbol{p}_5)\cdot(\boldsymbol{p}_j - \boldsymbol{p}_5)\right]_{1\leq i,j\leq 4} =$$
$$= \begin{bmatrix} \Delta_{1,5} & \Delta_{1,5}+\Delta_{2,5}-\Delta_{1,2} & \Delta_{1,5}+\Delta_{3,5}-\Delta_{1,3} & \Delta_{1,5}+\Delta_{4,5}-\Delta_{1,4} \\ \Delta_{1,5}+\Delta_{2,5}-\Delta_{1,2} & \Delta_{2,5} & \Delta_{2,5}+\Delta_{3,5}-\Delta_{2,3} & \Delta_{2,5}+\Delta_{4,5}-\Delta_{2,4} \\ \Delta_{1,5}+\Delta_{3,5}-\Delta_{1,3} & \Delta_{2,5}+\Delta_{3,5}-\Delta_{2,3} & \Delta_{3,5} & \Delta_{3,5}+\Delta_{4,5}-\Delta_{3,4} \\ \Delta_{1,5}+\Delta_{4,5}-\Delta_{1,4} & \Delta_{2,5}+\Delta_{4,5}-\Delta_{2,4} & \Delta_{3,5}+\Delta_{4,5}-\Delta_{3,4} & \Delta_{4,5} \end{bmatrix}.$$

The rank of A is ≤ 3, so the rank of $2A^TA$ is ≤ 3 and the determinant of the matrix above is 0 which gives a relation among the $\Delta_{i,j}$. The determinant is of the form $c_2\Delta_{4,5}^2 + c_1\Delta_{4,5} + c_0$, where c_0, c_1, c_2 are polynomials in $\Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4}, \Delta_{1,5}, \Delta_{2,3}, \Delta_{2,4}, \Delta_{2,5}, \Delta_{3,4}, \Delta_{3,5}$. If $c_2(\mathcal{M}) \neq 0$ then there are at most 2 distinct possible values for $\Delta_{4,5}$ if the values of the other fundamental polynomial invariants $\Delta_{i,j}$ are fixed. Geometrically, $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4$ and $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_5$ form two tetrahedra that are fixed up to translations, rotations and reflections. Depending on the orientations of the two tetrahedra, there are 2 possible values for $\Delta_{4,5}$. If $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ all lie on some line ℓ then rotating $\boldsymbol{p}_4$ and $\boldsymbol{p}_5$ about this line will only change the distance $\delta_{4,5}$. In this case, there are infinitely many values for $\Delta_{4,5}$ and $c_2(\mathcal{M}) = c_1(\mathcal{M}) = c_0(\mathcal{M}) = 0$. This example shows, that in some computer vision applications we may get too many fundamental polynomial invariants.

Rational Invariants

Sometimes there are no non-trivial polynomial invariants. However, there may still be rational functions that are invariant under the group action. If a group G acts on $\mathbb{R}^d$, then the rational functions on $\mathbb{R}^d$ form a field $\mathbb{R}(x_1, x_2, \ldots, x_d)$ with an action of G on it. If $f(x_1, \ldots, x_d) = a(x_1, \ldots, x_d)/b(x_1, \ldots, x_d)$ with $a(x_1, x_2, \ldots, x_d), b(x_1, x_2, \ldots, x_d)$ in the polynomial ring $\mathbb{R}[x_1, x_2, \ldots, x_d]$ and $\mathcal{M} \in \mathbb{R}^n$, then $f(\mathcal{M})$ is only defined when $b(\mathcal{M})$ is nonzero. If $f_1, f_2, \ldots, f_r \in \mathbb{R}(x_1, x_2, \ldots, x_d)$ are rational functions, then $\mathbb{R}(f_1, f_2, \ldots, f_r)$ is defined as the set of all $a(f_1, f_2, \ldots, f_r)/b(f_1, f_2, \ldots, f_r)$ where a and b are polynomials in r variables such that $b(f_1, f_2, \ldots, f_r) \neq 0$. Now $\mathbb{R}(f_1, f_2, \ldots, f_r)$ is the subfield of $\mathbb{R}(x_1, x_2, \ldots, x_d)$ generated by $f_1, f_2, \ldots, f_r$. A rational function $f \in \mathbb{R}(x_1, x_2, \ldots, x_d)$ is G-invariant if $f(g\cdot\mathcal{M}) = f(\mathcal{M})$ for all $g \in G$ and all $\mathcal{M}$ for which $f(g\cdot\mathcal{M})$ and $f(\mathcal{M})$ are defined. The set of all G-invariant rational functions form a subfield $\mathbb{R}(x_1, x_2, \ldots, x_d)^G$ of $\mathbb{R}(x_1, x_2, \ldots, x_d)$. We will call $f_1, f_2, \ldots, f_r \in \mathbb{R}(x_1, x_2, \ldots, x_d)^G$ a system of fundamental rational invariants if every rational invariant is a rational expression in $f_1, f_2, \ldots, f_r$, i.e., if $f_1, f_2, \ldots, f_r$ generate the field of rational invariants $\mathbb{R}(x_1, x_2, \ldots, x_d)^G$. It is known that every subfield of $\mathbb{R}(x_1, x_2, \ldots, x_d)$ has finitely many field generators. So there always exist finitely many fundamental rational invariants. Often one can choose fundamental rational invariants that do not have any algebraic relations.

Instead of the action of $\mathrm{E}(3)$, consider the action of the non-compact group $\mathrm{GL}(3)$ on models $\mathcal{M} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_d)$. For this group action, let $g(t) = t\cdot\mathrm{id}$. Then we have $g(t)\cdot\mathcal{M} = t\cdot\mathcal{M}$ and $\lim_{t\to 0} g(t)\cdot\mathcal{M} =$

$(\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0})$. If f is a polynomial invariant then $f(\mathcal{M}) = f(g(t) \cdot \mathcal{M})$. Using the continuity of f, we get $f(\mathcal{M}) = \lim_{t\to 0} f(g(t) \cdot \mathcal{M}) = f(\lim_{t\to 0} g(t) \cdot \mathcal{M}) = f(\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0})$. This shows that f is a constant polynomial. So there are no interesting polynomial invariants. However, there exist non-trivial rational invariants. Suppose that $d \geq 4$ and let us consider models $\mathcal{M} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_d)$ for which $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ are linearly independent. For $j = 4, 5, \ldots, d$ we can write $\boldsymbol{p}_j = \lambda_{j,1}\boldsymbol{p}_1 + \lambda_{j,2}\boldsymbol{p}_2 + \lambda_{j,3}\boldsymbol{p}_3$. Up to the GL(3)-action, the model $\mathcal{M}$ is completely determined by the $3(d-3) = 3d-9$ parameters $\lambda_{j,i}$ with $1 \leq i \leq 3$ and $4 \leq j \leq d$. The $\lambda_{j,i}$ are rational invariants, namely

$$\lambda_{j,1} = \frac{\det\begin{bmatrix}\boldsymbol{p}_j & \boldsymbol{p}_2 & \boldsymbol{p}_3\end{bmatrix}}{\det\begin{bmatrix}\boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3\end{bmatrix}}, \quad \lambda_{j,2} = \frac{\det\begin{bmatrix}\boldsymbol{p}_1 & \boldsymbol{p}_j & \boldsymbol{p}_3\end{bmatrix}}{\det\begin{bmatrix}\boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3\end{bmatrix}}, \quad \lambda_{j,3} = \frac{\det\begin{bmatrix}\boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_j\end{bmatrix}}{\det\begin{bmatrix}\boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3\end{bmatrix}}.$$

These $3d-9$ invariants form a system of fundamental rational invariants, and they are algebraically independent.

Algebraic Invariants

A continuous function $f(x_1, x_2, \ldots, x_d)$ defined on some open connected subset $U \subseteq \mathbb{R}^d$ is called algebraic if there is a positive integer m and there exist rational functions $h_0, h_1, \ldots, h_{m-1} \in \mathbb{R}(x_1, x_2, \ldots, x_d)$ such that

$$f^m + h_{m-1}f^{m-1} + \cdots + h_1 f + h_0 = 0. \tag{4}$$

Suppose that G is an group acting on $\mathbb{R}^n$. We say that f is an *algebraic invariant* if we can choose $h_0, h_1, \ldots, h_{m-1}$ to be rational invariants. If $f \in \mathbb{R}(x_1, x_2, \ldots, x_d)$ is a rational function and an algebraic invariant, and the group G is a connected Lie group, then f is actually a rational invariant.

COUNTING THE NUMBER OF INDEPENDENT INVARIANTS

In spacecraft navigation (and in computer vision) it is often important to know the maximum number of functionally independent invariants. This task requires some care and is not always straightforward. However, the ability to enumerate the number of functionally independent invariants is essential for the proper design of an invariant-based object recognition pipeline. There are numerous examples of algorithms that unknowingly use less than the number of independent invariants (and so there is useful information that is left unused) or that use more than the number of independent invariants (and so some of the information is actually redundant). Thus, establishing the existence of invariants and enumerating them are amongst the first steps in the rigorous development of an any algorithm that uses invariants.

Transcendence Degree

We saw above that is often convenient to work with rational invariants (e.g., instead of polynomial invariants). Thus, if we consider rational invariants, then the number of independent invariants relates to the notion of transcendence degree of a field extension. For more details on transcendence degree and field extensions, see [17, Chapter VIII] and [18, Chapter VI]. Suppose that $\mathbb{R} \subseteq K \subseteq L$ are fields. We say that $f_1, f_2, \ldots, f_r \in L$ are algebraically dependent over K if there exists a nonzero polynomial h with coefficients in K such that $h(f_1, f_2, \ldots, f_r) = 0$. The transcendence degree of L over K, denoted by $\operatorname{trdeg}(L/K)$, is the supremum over all nonnegative integers r for which there exists $f_1, f_2, \ldots, f_r \in L$ that are algebraically independent over K. Suppose that $f_1, f_2, \ldots, f_r \in L$ are algebraically independent over K and r is maximal, i.e., $r = \operatorname{trdeg}(L/K)$. Then we call $f_1, f_2, \ldots, f_r$ a transcendence basis of L over K. The field $K(f_1, f_2, \ldots, f_r)$ is a subfield of L, but they may not be equal. However, every element of L is algebraic over $K(f_1, f_2, \ldots, f_r)$ and by the Primitive Element Theorem (see [17, Theorem V.4.6]), there exists an $f_{r+1} \in L$ such that $L = K(f_1, f_2, \ldots, f_{r+1})$. Moreover, if $K \subseteq L \subseteq M$ is a chain of field extensions, then we have (see [18, Theorem VI.1.11])

$$\operatorname{trdeg}(M/K) = \operatorname{trdeg}(M/L) + \operatorname{trdeg}(L/K). \tag{5}$$

The transcendence degree of the rational function field $\mathbb{R}(x_1, x_2, \ldots, x_d)$ over $\mathbb{R}$ is d. In particular, if $f_1, f_2, \ldots, f_r \in \mathbb{R}(x_1, x_2, \ldots, x_d)$ and $r > d$, then $f_1, f_2, \ldots, f_r$ must be algebraically dependent over $\mathbb{R}$.

Suppose that $f_1, f_2, \ldots, f_r \in \mathbb{R}(x_1, x_2, \ldots, x_d)$ are possibly algebraically dependent and consider the field $K = \mathbb{R}(f_1, f_2, \ldots, f_r)$. There is a easy way to find the transcendence degree of K over $\mathbb{R}$. Consider the $r \times d$ Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_d} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_d} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x_1} & \frac{\partial f_r}{\partial x_2} & \cdots & \frac{\partial f_r}{\partial x_d} \end{bmatrix}.$$

The rank of J, as a matrix with entries in the field $\mathbb{R}(x_1, x_2, \ldots, x_d)$ (or K) is exactly equal to $\operatorname{trdeg}(K/\mathbb{R})$. For example, suppose that $f_1 = x_1 + x_2 + x_3$, $f_2 = x_1^2 + x_2^2 + x_3^2$ and $f_3 = x_1^3 + x_2^3 + x_3^3$. Then we have

$$J = \begin{bmatrix} 1 & 1 & 1 \\ 2x_1 & 2x_2 & 2x_3 \\ 3x_1^2 & 3x_2^2 & 3x_3^2 \end{bmatrix}.$$

Evaluating at $\mathcal{M} = [-1\ 0\ 1]^T$ gives the matrix

$$J(\mathcal{M}) = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & 2 \\ 3 & 0 & 3 \end{bmatrix}$$

whose has determinant is equal to 12. Consequently, we find that f_1, f_2, f_3 are algebraically independent since the determinant of J is nonzero.

Invariants and Transcendence Degree

Suppose that G is a group acting on $\mathbb{R}^d$. By the *number of independent rational invariants* we mean the transcendence degree $r = \operatorname{trdeg}(\mathbb{R}(x_1, x_2, \ldots, x_d)^G/\mathbb{R})$. We can choose a transcendence basis $f_1, f_2, \ldots, f_r$ of $\mathbb{R}(x_1, x_2, \ldots, x_d)^G$ over $\mathbb{R}$. Now every rational invariant $\mathbb{R}(x_1, x_2, \ldots, x_d)^G$ is an algebraic function in $f_1, f_2, \ldots, f_r$. If $\mathbb{R}(x_1, x_2, \ldots, x_d)^G$ is not equal to $\mathbb{R}(f_1, f_2, \ldots, f_r)$ then we can choose a rational invariant f_{r+1} such that $\mathbb{R}(x_1, x_2, \ldots, x_d)^G = \mathbb{R}(f_1, f_2, \ldots, f_{r+1})$. So the number of fundamental rational invariants we choose is equal, or 1 more than the number of independent rational invariants.

Suppose that G is an *algebraic* group acting on $\mathbb{R}^d$. This means that G is isomorphic to a subgroup of $\mathrm{GL}_n(\mathbb{R})$ for some n that is defined by polynomial equations. Let $G \cdot \mathcal{M}$ be the orbit of $\mathcal{M}$ and $G_{\mathcal{M}} = \{g \in G \mid g \cdot \mathcal{M} = \mathcal{M}\}$ be the stabilizer subgroup. Then we have (see e.g. [19, §1.4])

$$\dim G = \dim G \cdot \mathcal{M} + \dim G_{\mathcal{M}}. \tag{6}$$

For a random $\mathcal{M} \in \mathbb{R}^n$, if the dimension of $G \cdot \mathcal{M}$ is maximal, then the dimension of the stabilizer $G_{\mathcal{M}}$ is minimal. It follows from [19, §2.4] and (5) that for this random $\mathcal{M}$ we have

$$\dim G \cdot \mathcal{M} = \operatorname{trdeg}(\mathbb{R}(x_1, \ldots, x_d)/\mathbb{R}(x_1, \ldots, x_d)^G) = d - \operatorname{trdeg}(\mathbb{R}(x_1, \ldots, x_d)^G/\mathbb{R}). \tag{7}$$

We can use this formula to count the number of independent invariants. Consider again the example of the group $G = \mathrm{E}(3)$ on models $\mathcal{M} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_d) \in (\mathbb{R}^3)^d \cong \mathbb{R}^{3d}$ consisting of d points. The group $\mathrm{E}(3)$ has dimension 6. If $d = 1$, then the stabilizer of any $\mathcal{M} = (\boldsymbol{p}_1)$ is $\mathrm{O}(3)$ which has dimension 3. For $d = 2$, and $\mathcal{M} = (\boldsymbol{p}_1, \boldsymbol{p}_2)$ is randomly chosen, then $\boldsymbol{p}_1 \neq \boldsymbol{p}_2$ and the stabilizer is isomorphic to $\mathrm{O}(1)$ (and consists of rotations about the line through $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ and reflections in planes through this line). So the stabilizer has dimension 1 in that case. For $d = 3$ and $\mathcal{M} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3)$ randomly chosen, $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ are distinct and do not lie on a line. The stabilizer in this case has only 2 elements (identity and reflection in the plane through $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$) and has dimension 0. For $d \geq 4$ the stabilizer of a random $\mathcal{M}$ is trivial and has dimension 0.

These results may now be used to find the number of independent invariants $r = \operatorname{trdeg}(\mathbb{R}(x_1, y_1, z_1, \ldots, x_d, y_d, z_d)^{E(3)}/\mathbb{R})$. Specifically, from Eq. (7) we have that $\dim G \cdot \mathcal{M} = 3d - r$. Moreover, since $\dim G = 6$ for $\mathrm{E}(3)$, we may rearrange Eq. (7) to find that

$$6 - \dim G_{\mathcal{M}} = \dim G - \dim G_{\mathcal{M}} = \dim G \cdot \mathcal{M} = 3d - r,$$

so we have $r = 3d - 6 + \dim G_{\mathcal{M}}$. This results in the following number of independent invariants r for different numbers of d points

$$\begin{array}{c||c|c|c|} d & \dim G_{\mathcal{M}} & \dim G \cdot \mathcal{M} & r \\ \hline 1 & 3 & 3 & 0 \\ 2 & 1 & 5 & 1 \\ \geq 3 & 0 & 6 & 3d-6 \end{array}. \tag{8}$$

Derivations and Subfields

Before discussing counting view invariants, we need to understand derivations on function fields. See [17, Chapter VIII] for more details on derivations and field extensions. Suppose that $\mathbb{R} \subseteq K \subseteq L$ are fields. A *derivation* of L over K is a function $D : L \to L$ that satisfies the Leibniz' rule: $D(ab) = aD(b) + D(a)b$ for all $a, b \in L$, and $D(a) = 0$ for all $a \in K$. One can verify that a derivation $D \in \mathrm{Der}(L/K)$ is a K-linear map $L \to L$. We write $\mathrm{Der}(L/K)$ for all derivations of L over K. Now $\mathrm{Der}(L/K)$ is closed under addition. Also, if $\lambda \in L$ and $D \in \mathrm{Der}(L/K)$ then λD again lies in $\mathrm{Der}(L/K)$. So $\mathrm{Der}(L/K)$ is an L-vector space. One has (see [17, Proposition VIII.5.5])

$$\dim \mathrm{Der}(L/K) = \mathrm{trdeg}(L/K). \tag{9}$$

Moreover, if $D, E \in \mathrm{Der}(L/K)$ then $[D, E] = DE - ED : L \to L$ again lies in $\mathrm{Der}(L/K)$. This makes $\mathrm{Der}(L/K)$ into a Lie algebra. For example, if $L = \mathbb{R}(x_1, x_2, \ldots, x_d)$, then the partial derivatives $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_d}$ are derivations of L over $\mathbb{R}$. In fact, they form a basis of $\mathrm{Der}(\mathbb{R}(x_1, x_2, \ldots, x_d)/\mathbb{R})$ as an $\mathbb{R}(x_1, x_2, \ldots, x_d)$-vector space.

Suppose that G is an algebraic group acting on $\mathbb{R}^d$. Let $L = \mathbb{R}(x_1, x_2, \ldots, x_d)$. We can find $\mathrm{Der}(L/L^G)$ as follows. The algebraic group G is a Lie group with some Lie algebra $\mathfrak{g}$. To an element $A \in \mathfrak{g}$ we can associate a derivation $D = \phi(A)$ on $L = \mathbb{R}(x_1, x_2, \ldots, x_d)$ as follows. We can choose a curve $g : (-\varepsilon, \varepsilon) \to G$ such that $g(0)$ is the identity element and $A = g'(0)$. Then we have

$$D(f) = \left. \frac{d\, f(g(t))}{d\, t} \right|_{t=0}.$$

We have that $\mathrm{Der}(\mathbb{R}(x_1, \ldots, x_d)/\mathbb{R}(x_1, x_2, \ldots, x_d)^G)$ is the L-vector space spanned by $\phi(\mathfrak{g})$. To illustrate this, consider again the action of $G = \mathrm{E}(3)$ on models $\mathcal{M} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_d)$ consisting of d points. The group G acts on $L = \mathbb{R}(x_1, y_1, z_1, \ldots, x_d, y_d, z_d)$. Infinitesemal translations in the x-, y- and z-direction correspond to the derivations

$$D_1 = \sum_{i=1}^{d} \frac{\partial}{\partial x_i}, \quad D_2 = \sum_{i=1}^{d} \frac{\partial}{\partial y_i}, \quad D_3 = \sum_{i=1}^{d} \frac{\partial}{\partial z_i}. \tag{10}$$

Infinitesemal rotations about the x-, y- and z-axes correspond to the derivations

$$D_4 = \sum_{i=1}^{d} z_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial z_i}, \quad D_5 = \sum_{i=1}^{d} z_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial z_i}, \quad D_6 = \sum_{i=1}^{d} y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i}. \tag{11}$$

Now $\mathrm{Der}(L/L^G)$ is exactly the L-vector space spanned by $D_1, D_2, D_3, D_4, D_5, D_6$. The dimension of $\mathrm{Der}(L/L^G)$ is equal to $\dim G \cdot \mathcal{M}$ (see 7). In particular, $D_1, D_2, D_3, D_4, D_5, D_6$ are linear independent if and only if $d \geq 3$.

Suppose that K_1, K_2 are subfields of L containing $\mathbb{R}$. If $K_1 \subseteq K_2$, then we have $\mathrm{Der}(L/K_2) \subseteq \mathrm{Der}(L/K_1)$. In general, we have that the L-vector spaces $\mathrm{Der}(L/K_1)$ and $\mathrm{Der}(L/K_2)$ are contained in $\mathrm{Der}(L/(K_1 \cap K_2))$. Let $\langle \mathrm{Der}(L/K_1), \mathrm{Der}(L/K_2) \rangle$ be the Lie algebra generated by $\mathrm{Der}(L/K_1)$ and $\mathrm{Der}(L/K_2)$. Then we have

$$\langle \mathrm{Der}(L/K_1), \mathrm{Der}(L/K_2) \rangle \subseteq \mathrm{Der}(L/(K_1 \cap K_2)) \tag{12}$$

It was shown in [20] that we have equality if K_1, K_2, L are algebraically closed fields. This also implies that we have equality if K_1 and K_2 are algebraically closed within L.

As an example, we compute $K_1 \cap K_2$ where $K_1 = \mathbb{R}(x, y+z^2)$, $K_2 = \mathbb{R}(y, z+x^2)$ of $L = \mathbb{R}(x,y,z)$. A derivation $D \in \mathrm{Der}(L/\mathbb{R})$ is of the form $a(x,y,z)\frac{\partial}{\partial x} + b(x,y,z)\frac{\partial}{\partial y} + c(x,y,z)\frac{\partial}{\partial z}$. If $D \in \mathrm{Der}(L/K_1)$, then $a(x,y,z) = D(x) = 0$ and $D(y+z^2) = b(x,y,z) + 2zc(x,y,z)$. So $D = c(x,y,z)(-2z\frac{\partial}{\partial y} + \frac{\partial}{\partial z})$. So $\mathrm{Der}(L/K_1)$ is the L-vector space spanned by the derivation $D_1 = -2z\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. Similarly, $\mathrm{Der}(L/K_2)$ is the L-vector space spanned by $D_2 = \frac{\partial}{\partial x} - 2x\frac{\partial}{\partial z}$. We compute $[D_1, D_2]$ as follows. From $D_1D_2(x) = D_1(1) = 0$ and $D_2D_1(x) = D_2(0) = 0$ follows that $[D_1, D_2](x) = 0$. Similarly, $[D_1, D_2](y) = D_1D_2(y) - D_2D_1(y) = 0 - 4x = -4x$ and $[D_1, D_2](z) = 0$. So we have $D_3 := [D_1, D_2] = -4x\frac{\partial}{\partial y}$. It is clear that $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ lie in the L-span of D_1, D_2, D_3. So $\mathrm{Der}(L/(K_1 \cap K_2))$ is equal to $\mathrm{Der}(L/\mathbb{R})$ which has dimension 3. So we have $\mathrm{trdeg}(L/(K_1 \cap K_2)) = \dim \mathrm{Der}(L/(K_1 \cap K_2) = 3$ and

$$\mathrm{trdeg}(K_1 \cap K_2/\mathbb{R}) = \mathrm{trdeg}(L/\mathbb{R}) - \mathrm{trdeg}(L/(K_1 \cap K_2)) = 3 - 3 = 0.$$

This implies that $K_1 \cap K_2 = \mathbb{R}$.

Counting Independent View Invariants

If we consider view invariants rather than actual invariants, then we have to use different methods to find the number of independent invariants. As it turns out, we can find the view invariants as an intersection of fields.

We have already shown that there are no view invariants for orthogonal projections for point clouds in $\mathbb{R}^3$. We will now show that there are no view invariants if we use a pinhole camera with a perspective projection. While this fact has been known for some time [21, 22, 23], it is instructive to develop this familiar result with the present theory as an illustrative example. Let $\pi : \mathbb{R}^3 \setminus \{0\} \to \mathbb{P}^2$ be the perspective projection. If we use projective coordinates $[x : y : z] \in \mathbb{P}^2$, then the rational functions on $\mathbb{P}^2$ form the field $\mathbb{R}(\frac{x}{z}, \frac{y}{z})$, which is a subfield of the field $\mathbb{R}(x,y,z)$ of rational functions on $\mathbb{R}^3$. Now $\mathrm{Der}(\mathbb{R}(x,y,z)/\mathbb{R}(\frac{x}{z}, \frac{y}{z}))$ is the $\mathbb{R}(x,y,z)$ spanned by the Euler derivation $E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$. If we have a model $\mathcal{M} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_d) \in (\mathbb{R}^3)^d$ of d (nonzero) points, then the image $\mathcal{I} = \pi(\mathcal{M})$. The projection π gives an inclusion of fields

$$K = \mathbb{R}\Big(\frac{x_1}{z_1}, \frac{y_1}{z_1}, \ldots, \frac{x_d}{z_d}, \frac{y_d}{z_d}\Big) \subseteq L = \mathbb{R}(x_1, y_1, z_1, \ldots, x_d, y_d, z_d).$$

The L-vector space $\mathrm{Der}(L/K)$ is spanned by the derivations $E_i = x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i} + z_i\frac{\partial}{\partial z_i}$, $i = 1, 2, \ldots, d$. We now consider the action of $G = \mathrm{E}(3)$ on these models. The view invariants are exactly those invariants that lie in the field K. In other words, we want to find the intersection field $K \cap L^G$. Now $\mathrm{Der}(L/(K \cap L^G))$ contains the Lie algebra generated by $D_1, D_2, \ldots, D_6 \in \mathrm{Der}(L/L^G)$ and $E_1, E_2, \ldots, E_d \in \mathrm{Der}(L/K)$. We have $[D_1, E_i] = \frac{\partial}{\partial x_i}$, $[D_2, E_i] = \frac{\partial}{\partial y_i}$ and $[D_3, E_i] = \frac{\partial}{\partial z_i}$ for $i = 1, 2, \ldots, d$. So $\mathrm{Der}(L/(K \cap L^G))$ contains the basis of $\mathrm{Der}(L/\mathbb{R})$. We conclude that $\mathrm{Der}(L/\mathbb{R}) = \mathrm{Der}(L/(K \cap L^G))$, $\mathrm{trdeg}(K \cap L^G/\mathbb{R}) = 0$ and $K \cap L^G = \mathbb{R}$. This proves that there are no view invariants. Importantly, we have not used the derivations D_4, D_5, D_6. So the same argument shows that there are no view invariants for this perspective projection if we use a smaller symmetry group $G \cong \mathbb{R}^3$ consisting of just translations. This means, that if we use a perspective camera with fixed (known) orientation but arbitrary location, then there still are no view invariants. Interestingly, however, if we constrain the observer motion to a known line then view invariants are introduced—a fact which was proven using the approach shown here in Ref. [24].

The situation radically changes if we assume that the points $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_d$ are coplanar with $d \geq 4$. We write $\boldsymbol{p}_i = [x_i \; y_i \; z_i]^T$. The coordinates of $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_d$ form a field $L = \mathbb{R}(x_1, y_1, z_1, \ldots, x_d, y_d, z_d)$, but now there are relations among the coordinate functions x_i, y_i, z_i, $1 \leq i \leq d$. And $\mathrm{trdeg}(L/\mathbb{R})$ is strictly smaller than d. Let us consider coplanar models $\mathcal{M} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_d)$ where $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_d$ span $\mathbb{R}^3$ (but are oplanar). Then the plane through these points is given by an equation $ax + by + cz = 1$. Now $\mathcal{M}$ is parameterized by the variables a, b, c and $u_i = \frac{x_i}{z_i}, v_i = \frac{y_i}{z_i}$ for $i = 1, 2, \ldots, d$. These parameters form a field

$$L = \mathbb{R}\Big(a, b, c, u_1, v_1, u_2, v_2, \ldots, u_d, v_d\Big).$$

We can express x_i, y_i, z_i in these field generators:

$$x_i = \frac{u_i}{au_i + bv_i + c}, \quad y_i = \frac{v_i}{au_i + bv_i + c}, \quad z_i = \frac{1}{au_i + bv_i + c}.$$

Let us investigate the action of G on the field L. If we translate in the x-direction over t, we get $x_i(t) = x_i + t$, $y_i(t) = y_i$ and $z_i(t) = z_i$. The equation $ax + by + cz = 1$ of the plane changes to $a(x+t) + by + cz = 1 + at$ or equivalently

$$(\frac{a}{a+t})x + (\frac{b}{a+t})y + (\frac{c}{a+t})z = 1.$$

So we get $a(t) = \frac{a}{1+at}$, $b(t) = \frac{b}{1+at}$ and $c(t) = \frac{c}{1+ct}$. We also get

$$u_i(t) = \frac{x_i(t)}{z_i(t)} = \frac{x_i + t}{z_i} = \frac{\frac{u_i}{au_i+bv_i+c} + t}{\frac{1}{au_i+bv_i+c}} = u_i + t(au_i + bv_i + c)$$

and $v_i(t) = \frac{y_i(t)}{z_i(t)} = \frac{x_i}{z_i} = v_i$. We get the infinitesemal action by differentiating t, and setting $t = 0$. This gives the following derivation:

$$D_1 = \sum_{i=1}^{d}(au_i + bv_i + c)\frac{\partial}{\partial u_i} - a\Big(a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c}\Big).$$

By symmetry, translation in the y-direction gives

$$D_2 = \sum_{i=1}^{d}(au_i + bv_i + c)\frac{\partial}{\partial v_i} - b\Big(a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c}\Big).$$

Infinitesemal translation in the z direction is slightly different, but similar computations show that we get the derivation

$$D_3 = \sum_{i=1}^{d}(au_i + bv_i + c)\Big(u_i\frac{\partial}{\partial u_i} + v_i\frac{\partial}{\partial v_i}\Big) - c\Big(a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c}\Big).$$

Rotations in 3 directions give three more derivations D_4, D_5, D_6, and $\mathrm{Der}(L/L^G)$ is spanned by the derivations $D_1, D_2, \dots, D_6$ as an L-vector space.

The projection $\pi : \mathbb{R}^3 \setminus \{0\} \to \mathbb{P}^2$ corresponds to the inclusion of the field

$$K = \mathbb{R}(u_1, v_1, u_2, v_2, \dots, u_d, v_d)$$

into L. It is clear that $\mathrm{Der}(L/K)$ is the spanned by the derivations $\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}$. The Lie algebra $\mathrm{Der}(L/(K \cap L^G))$ contains the derivations $D_1, D_2, \dots, D_6, \frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}$. In particular, it contains

$$D_1' = \sum_{i=1}^{d}(au_i + bv_i + c)\frac{\partial}{\partial u_i}, \quad D_2' = \sum_{i=1}^{d}(au_i + bv_i + c)\frac{\partial}{\partial v_i}, \quad D_3' = \sum_{i=1}^{d}(au_i + bv_i + c)\Big(u_i\frac{\partial}{\partial u_i} + v_i\frac{\partial}{\partial v_i}\Big).$$

So it also contains

$$E_1 = \Big[\frac{\partial}{\partial a}, D_1'\Big] = \sum_{i=1}^{d} u_i\frac{\partial}{\partial u_i}, \quad E_2 = \Big[\frac{\partial}{\partial b}, D_1'\Big] = \sum_{i=1}^{d} v_i\frac{\partial}{\partial u_i}, \quad E_3 = \Big[\frac{\partial}{\partial c}, D_1'\Big] = \sum_{i=1}^{d}\frac{\partial}{\partial u_i},$$

$$E_4 = \Big[\frac{\partial}{\partial a}, D_2'\Big] = \sum_{i=1}^{d} u_i\frac{\partial}{\partial v_i}, \quad E_5 = \Big[\frac{\partial}{\partial b}, D_2'\Big] = \sum_{i=1}^{d} v_i\frac{\partial}{\partial v_i}, \quad E_6 = \Big[\frac{\partial}{\partial c}, D_2'\Big] = \sum_{i=1}^{d}\frac{\partial}{\partial v_i},$$

$$E_7 = \Big[\frac{\partial}{\partial a}, D_3'\Big] = \sum_{i=1}^{d}\sum_{i=1}^{d} u_i\Big(u_i\frac{\partial}{\partial u_i} + v_i\frac{\partial}{\partial v_i}\Big), \quad E_8 = \Big[\frac{\partial}{\partial b}, D_3'\Big] = \sum_{i=1}^{d}\sum_{i=1}^{d} v_i\Big(u_i\frac{\partial}{\partial u_i} + v_i\frac{\partial}{\partial v_i}\Big).$$

For $d \geq 4$, the derivations $\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}, E_1, E_2, \ldots, E_8$ are independent over L. To see this, it suffices to show that $E_1, E_2, \ldots, E_8$ are independent. We consider the matrix A whose rows are

$$\begin{bmatrix} E_i(u_1) & E_i(u_2) & E_i(u_3) & E_i(u_4) & E_i(v_1) & E_i(v_2) & E_i(v_3) & E_i(v_4) \end{bmatrix}$$

for $i = 1, 2, \ldots 8$. We have

$$A = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & 0 & 0 & 0 & 0 \\ v_1 & v_2 & v_3 & v_4 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_1 & u_2 & u_3 & u_4 \\ 0 & 0 & 0 & 0 & v_1 & v_2 & v_3 & v_4 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ u_1^2 & u_2^2 & u_3^2 & u_4^2 & u_1 v_1 & u_2 v_2 & u_3 v_3 & u_4 v_4 \\ u_1 v_1 & u_2 v_2 & u_3 v_3 & u_4 v_4 & v_1^2 & v_2^2 & v_3^2 & v_4^2 \end{bmatrix}.$$

This matrix has rank 8 which can be checked by plugging in random numbers for $u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4$ and verifying that the determinant is nonzero. This means that the rows of A are linearly independent over L and therefore $E_1, E_2, \ldots, E_8$ are linearly independent over L. It follows now that the dimension of $\mathrm{Der}(L/(K \cap L^G))$ is at least $3 + 8 = 11$, so $\mathrm{trdeg}(L/(K \cap L^G)) \geq 11$. Now $\mathrm{trdeg}(K \cap L^G)/\mathbb{R}) = \mathrm{trdeg}(L/\mathbb{R}) - \mathrm{trdeg}(L/(K \cap L^G)) \leq (2d+3) - 11 = 2d - 8$. So there are at most $2d - 8$ independent view invariants. As we will see later, there are exactly $2d - 8$ view invariants. Note that $\mathrm{GL}(3)$ acts on $\mathbb{R}^3$ and therefore on the field $\mathbb{R}(x_1, y_1, z_1, \ldots, x_d, y_d, z_d)$. This restricts to an action of $\mathrm{GL}(3)$ on the subfield K. Because multiples of the identity act trivially on K, we have an action of $\mathrm{PGL}(3)$ on K. The action of the Lie algebra on K is exactly given by the derivations $E_1, E_2, \ldots, E_8$. So $K \cap L^G$ is contained in $K^{\mathrm{PGL}(3)}$ and they are infact equal.

INVARIANTS FOR 3D SENSORS

Consider here a 3D sensor (e.g., LIDAR [25]) that directly measures the sensor-relative location of 3D points within the observed scene. Assuming the observed objects are rigid, the transformation of model points to measurements is governed by the action of $G = \mathrm{SE}(3)$. When the model is described by a d-tuple of points, the invariants are the pairwise point distances (as described in earlier examples). When the model is described by a d-tuple of surfaces (e.g, planes) there are different invariants. These two situations are now discussed.

Set of 3D Points in General Position

Suppose we have objects whose models take the form $\mathcal{M} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_d)$, where the points $\{\boldsymbol{p}_i\}_{i=1}^d \in \mathbb{R}^3$ are in general position. Further suppose that we have a 3D sensor that measures the transformed location of these points $\{\boldsymbol{q}_i\}_{i=1}^d \in \mathbb{R}^3$ under the action of $\mathrm{E}(3)$ as described in Eq. (1). In this scenario, we have already shown [as summarized in Eq. (8)] that a d-tuple of $d \geq 3$ points in general position has $3d - 6$ independent invariants. One choice for these invariants are the pairwise distances $\delta_{i,j} = \|\boldsymbol{p}_i - \boldsymbol{p}_j\| = \|\boldsymbol{q}_i - \boldsymbol{q}_j\|$, but other choices are also possible. For example, since the pairwise distances are all invariant, one can use the law of cosines to show that the interior angles for every point triplet are also invariant. Thus, another choice for a set of independent invariants would be one of the pairwise distances (for scale) and $3d - 5$ interior angles (with no more than two angles from any triplet). Other combinations of independent distances and angles would also work. Moreover, any number of different invariant functions may be constructed (since any function of invariants is also invariant) and some invariant choices may be numerically preferable to others, though there will never be more than $3d - 6$ independent invariants.

The explicit use of pairwise distances between model points to recognize objects in LIDAR data has been used within the space community for some time, especially within the context of recognizing retroreflectors on the ISS [11]. Similar ideas have also been proposed to recognize naturally occurring keypoints found with 3D feature descriptors [13]. Moreover, if one is designing the object to be recognized (e.g., choosing the locations to place retroreflectors, fiducials, or other keypoints), then these same ideas may be used to design optimal patterns. For example, Ref. [7] describes how to choose patterns that maximize the difference in the pairwise distances, which makes pattern recognition easier. This approach is the theoretical basis for the perimeter reflector pattern chosen for the International Docking Adapter (IDA) standard [7].

Set of 3D Planes

Suppose we have objects whose models take the form $\mathcal{M} = (\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \dots, \boldsymbol{\pi}_d)$, where the planes $\{\boldsymbol{\pi}_i\}_{i=1}^d \in \mathbb{P}^3$ are in general position. Due to the duality of points and planes in $\mathbb{P}^3$, the number of invariants for a d-tuple of planes is the same as for a d-tuple of points. Thus, there is one invariant for a pair of planes and there are $3d-6$ independent invariants for $d \geq 3$ planes.

We will now briefly develop one set of independent invariants. Begin by recalling that a homogeneous point $\bar{\boldsymbol{p}} \in \mathbb{P}^3$ lies on the plane $\boldsymbol{\pi}_i \in \mathbb{P}^3$ when $\boldsymbol{\pi}_i^T \bar{\boldsymbol{p}} = 0$. Observe that the plane $\boldsymbol{\pi}_i$ may be described by a 4×1 vector parameterized as $\boldsymbol{\pi}_i \propto [\boldsymbol{n}_i^T, -\rho_i]$, where $\boldsymbol{n}_i \in \mathbb{R}^3$ is the plane unit normal and ρ_i is the perpendicular distance from the plane to the origin. Following Eq. (1), we describe the action of $\mathrm{E}(3)$ with the 4×4 matrix $\boldsymbol{H}$

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{r} \\ \boldsymbol{0}_{1\times 3} & 1 \end{bmatrix} \tag{13}$$

such that the homogeneous point $\bar{\boldsymbol{p}}$ is transformed according to $\bar{\boldsymbol{p}}' = \boldsymbol{H}\bar{\boldsymbol{p}}$. Therefore, $\boldsymbol{\pi}_i^T \boldsymbol{H}^{-1} \bar{\boldsymbol{p}}' = 0$ and the transformed plane is $\boldsymbol{\pi}_i' = \boldsymbol{H}^{-T}\boldsymbol{\pi}_i$. Thus, in the transformed frame, we once again have $\boldsymbol{\pi}_i'^T \bar{\boldsymbol{p}}' = 0$. Analytically inverting $\boldsymbol{H}$,

$$\boldsymbol{H}^{-1} = \begin{bmatrix} \boldsymbol{A}^T & -\boldsymbol{A}^T\boldsymbol{r} \\ \boldsymbol{0}_{1\times 3} & 1 \end{bmatrix} \tag{14}$$

allows us to directly compute $\boldsymbol{\pi}_i'$ as

$$\boldsymbol{\pi}_i' \propto \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0}_{3\times 1} \\ -\boldsymbol{r}^T\boldsymbol{A} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{n}_i \\ -\rho_i \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}\boldsymbol{n}_i \\ -\boldsymbol{r}^T\boldsymbol{A}\boldsymbol{n}_i - \rho_i \end{bmatrix} = \begin{bmatrix} \boldsymbol{n}_i' \\ -\rho_i' \end{bmatrix} \tag{15}$$

Thus, we see that the plane dihedral angles are invariants,

$$\cos\theta_{i,j}' = \boldsymbol{n}_i'^T \boldsymbol{n}_j' = \boldsymbol{n}_i^T \boldsymbol{A}^T \boldsymbol{A} \boldsymbol{n}_j = \boldsymbol{n}_i^T \boldsymbol{n}_j = \cos\theta_{i,j} \tag{16}$$

which makes use of the fact that $\boldsymbol{A}$ is an orthogonal matrix.

For $d = 2$ the only invariant is the single dihedral angle between the two planes. For $d = 3$, the three independent invariants are the three dihedral angles between each pair of planes. However, there are only $2d-3$ independent dihedral angles—such that there are $(3d-6)-(2d-3) = d-3$ other independent invariants that we must find for $d \geq 4$ planes. There are a variety of ways to compute to compute these additional invariants to form a complete set of independent invariants. For example, if we already have the first three planes $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \boldsymbol{\pi}_3$, then each additional plane for $d \geq 4$ comes with two new dihedral invariants (e.g., $\theta_{1,k}, \theta_{2,k}$) and a single new scaling invariant (e.g., $s_{1,2,3,k}$). Perhaps the easiest scaling invariant may be computed by recognizing that $\det(\boldsymbol{H}) = \pm 1$ since $\boldsymbol{H}$ describes the action of $\mathrm{E}(3)$. Therefore, one suitable scaling metric is

$$\begin{aligned} s_{1,2,3,k}' &= \| \det\left(\begin{bmatrix} \boldsymbol{\pi}_1' & \boldsymbol{\pi}_2' & \boldsymbol{\pi}_3' & \boldsymbol{\pi}_k' \end{bmatrix} \right) \| \\ &= \| \det\left(\boldsymbol{H}^{-T} \begin{bmatrix} \boldsymbol{\pi}_1 & \boldsymbol{\pi}_2 & \boldsymbol{\pi}_3 & \boldsymbol{\pi}_k \end{bmatrix} \right) \| \\ &= \| \det\left(\boldsymbol{H}^{-T} \right) \det\left(\begin{bmatrix} \boldsymbol{\pi}_1 & \boldsymbol{\pi}_2 & \boldsymbol{\pi}_3 & \boldsymbol{\pi}_k \end{bmatrix} \right) \| \\ &= \| \det\left(\begin{bmatrix} \boldsymbol{\pi}_1 & \boldsymbol{\pi}_2 & \boldsymbol{\pi}_3 & \boldsymbol{\pi}_k \end{bmatrix} \right) \| \\ &= s_{1,2,3,k} \end{aligned} \tag{17}$$

which is clearly the same both before and after transformation by $\boldsymbol{H}$. This invariant formulation assumes that the entries $\boldsymbol{\pi}_i$ and $\boldsymbol{\pi}_i'$ are scaled such that the first three elements have unity norm.

Invariants for a d-tuple of planes occurs often when viewing human-made objects, which are often constructed of planar facets. For example, a typical house has planes for each side of the building and inclined plans for each roof segment. Similar plane-based models are also useful for some spacecraft shapes.

VIEW INVARIANTS FOR CAMERAS

Consider here a perspective optical instrument (e.g., camera or telescope) that observes a 3D scene. In our introduction to invariant theory above, we have already seen that view invariants are a powerful tool for object recognition. However, we have also seen that view invariants do not always exist and that these are somewhat harder to construct than the invariants for a 3D sensor.

In this section, we explicitly develop the usual perspective camera model (i.e., the pinhole model). This model is then applied to a number of commonly encountered scenarios and view invariants are discussed. The theoretical framework presented in this manuscript is used to unify results developed within the computer vision and spaceflight communities over the past 50 years.

Camera Model

Assuming perspective projection, a calibrated camera measures the line-of-sight $\boldsymbol{\ell}_i \in \mathbb{P}^2$ describing the apparent direction to the camera-relative location $\boldsymbol{q}_i$. Thus, the pinhole camera model is given by

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \bar{\boldsymbol{x}}_i \propto \boldsymbol{\ell}_i \propto \boldsymbol{q}_i \tag{18}$$

where the 2D coordinate $[x_i, y_i]$ is where the LOS ray pierces the $z = 1$ plane (often called the image plane). A pixel location $[u_i, v_i]$ in a digital image are related to the image plane coordinate $[x_i, y_i]$ by an affine transformation $\boldsymbol{K}$,

$$\begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix} = \bar{\boldsymbol{u}}_i = \boldsymbol{K}\bar{\boldsymbol{x}}_i \tag{19}$$

where $\boldsymbol{K}$ is the so-called camera calibration matrix (see Ref. [26] for details). Therefore, the apparent pixel location of a camera-relative point $\boldsymbol{q}_i$ is given by

$$\bar{\boldsymbol{u}}_i \propto \boldsymbol{K}\boldsymbol{q}_i = \boldsymbol{K}\left(\boldsymbol{T}\boldsymbol{p}_i + \boldsymbol{r}\right) \tag{20}$$

Or, writing entirely in homogoneous coordinates, let $\bar{\boldsymbol{p}}_i^T = [\boldsymbol{p}_i^T, 1] \in \mathbb{P}^3$ such that

$$\bar{\boldsymbol{u}}_i \propto \boldsymbol{P}\bar{\boldsymbol{p}}_i \tag{21}$$

where

$$\boldsymbol{P} \propto \boldsymbol{K}\begin{bmatrix} \boldsymbol{T} & \boldsymbol{r} \end{bmatrix} \tag{22}$$

such that the camera model $\boldsymbol{P}$ has dimension describes the mapping $\pi : \mathbb{P}^3 \to \mathbb{P}^2$.

Planar Scenes

Consider the scenario of a camera viewing a planar (or nearly planar) scene. Such a scenario may occur when close to low-relief terrain or when viewing flat surfaces on many human-made objects. A number of unique and useful invariants arise as a result of constraining the observed features to lie in a plane—such that their projection into a digital image may be described by a homography. To see this, note that we may describe any point in the scene as $\boldsymbol{p}_i^T = [x_i, y_i, 0]$ without loss of generality. Or, equivalently,

$$\boldsymbol{p}_i = \boldsymbol{S}^T \begin{bmatrix} x_i \\ y_i \end{bmatrix} \tag{23}$$

where $\boldsymbol{S} = [\boldsymbol{I}_{2\times 2}, \boldsymbol{0}_{2\times 1}]$. Substituting into Eq. (22) yields the 3×3 homography matrix $\boldsymbol{H}$ with ambiguous scale

$$\boldsymbol{H} \propto \boldsymbol{K} \begin{bmatrix} \boldsymbol{T}\boldsymbol{S}^T & \boldsymbol{r} \end{bmatrix} = \boldsymbol{K} \begin{bmatrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 & \boldsymbol{r} \end{bmatrix} \tag{24}$$

where $\boldsymbol{T} = [\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3]$. The homography $\boldsymbol{H}$ is always full rank in practice since we assume that the camera location $\boldsymbol{r}$ does not lie in the same plane as the points (i.e., in the plane spanned by $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$). Thus, one has,

$$\begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix} \propto \boldsymbol{H} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} \tag{25}$$

It follows, therefore, that images of a planar scene are a mapping $\pi : \mathbb{P}^2 \to \mathbb{P}^2$ described by the action of G =PGL(3). Recognizing this allows for the straightforward development and enumeration of invariants.

Set of Coplanar Points

Suppose we have objects whose models take the form $\mathcal{M} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_d)$, where the points $\{\boldsymbol{p}_i\}_{i=1}^d \in \mathbb{R}^3$ are constrained to lie in the plane $\boldsymbol{\pi}_0$. Such a d-tuple of coplanar points is known to possess $2d - 8$ independent invariants for $d \geq 5$ points in general position on the plane (i.e., not colinear). This fact has been known for some time within the context of computer vision [27]. These invariants may be constructed through determinants, cross-ratios, transforming to a canonical pattern, or any number of other equivalent representations. A detailed development of invaraints for PGL(3) may be found in Refs. [4] and [27]. For example, in the case of $d = 5$, there are two independent invariants (corresponding to cross-ratios) that may be constructed from determinants:

$$I_1 = \frac{\det([\bar{\boldsymbol{x}}_2\ \bar{\boldsymbol{x}}_3\ \bar{\boldsymbol{x}}_1])\det([\bar{\boldsymbol{x}}_4\ \bar{\boldsymbol{x}}_5\ \bar{\boldsymbol{x}}_1])}{\det([\bar{\boldsymbol{x}}_2\ \bar{\boldsymbol{x}}_4\ \bar{\boldsymbol{x}}_1])\det([\bar{\boldsymbol{x}}_3\ \bar{\boldsymbol{x}}_5\ \bar{\boldsymbol{x}}_1])} = \frac{\det([\bar{\boldsymbol{u}}_2\ \bar{\boldsymbol{u}}_3\ \bar{\boldsymbol{u}}_1])\det([\bar{\boldsymbol{u}}_4\ \bar{\boldsymbol{u}}_5\ \bar{\boldsymbol{u}}_1])}{\det([\bar{\boldsymbol{u}}_2\ \bar{\boldsymbol{u}}_4\ \bar{\boldsymbol{u}}_1])\det([\bar{\boldsymbol{u}}_3\ \bar{\boldsymbol{u}}_5\ \bar{\boldsymbol{u}}_1])} \tag{26}$$

and

$$I_2 = \frac{\det([\bar{\boldsymbol{x}}_1\ \bar{\boldsymbol{x}}_3\ \bar{\boldsymbol{x}}_2])\det([\bar{\boldsymbol{x}}_4\ \bar{\boldsymbol{x}}_5\ \bar{\boldsymbol{x}}_2])}{\det([\bar{\boldsymbol{x}}_1\ \bar{\boldsymbol{x}}_4\ \bar{\boldsymbol{x}}_2])\det([\bar{\boldsymbol{x}}_3\ \bar{\boldsymbol{x}}_5\ \bar{\boldsymbol{x}}_2])} = \frac{\det([\bar{\boldsymbol{u}}_1\ \bar{\boldsymbol{u}}_3\ \bar{\boldsymbol{u}}_2])\det([\bar{\boldsymbol{u}}_4\ \bar{\boldsymbol{u}}_5\ \bar{\boldsymbol{u}}_2])}{\det([\bar{\boldsymbol{u}}_1\ \bar{\boldsymbol{u}}_4\ \bar{\boldsymbol{u}}_2])\det([\bar{\boldsymbol{u}}_3\ \bar{\boldsymbol{u}}_5\ \bar{\boldsymbol{u}}_2])} \tag{27}$$

where $\bar{\boldsymbol{x}}_i = [x_i, y_i, 1]^T \in \mathbb{P}^2$ is a homogeneous model point and $\bar{\boldsymbol{u}}_i = [u_i, v_i, 1]^T \in \mathbb{P}^2$ is a homogeneous image point. These exact equations may also be interpreted as a cross-ratio [4]. The center terms in Eqs. (26) and (27) are constructed directly from the model (i.e., only a function of $\{\bar{\boldsymbol{x}}_i\}_{i=1}^5$) and so they are a constant for any given model. Likewise, the right-hand terms are constructed from only measurements (i.e., only a function of $\{\bar{\boldsymbol{u}}_i\}_{i=1}^5$) and are always the same no matter the transformation $\boldsymbol{H}$ from Eq. (25). Finally, as discussed in Ref. [4], there are $5! = 120$ possible permutations of these five points leading to 30 distinct invariant values—only two of which are algebraically independent. In many cases (this one included) invariants may also be made permutation independent, as discussed in Refs. [5] and [28].

One of the critical observations is that there are no invariants for $d \leq 4$ coplanar points under the action of PGL(3). Consequently, the commonly pursued quest for invariants describing triplets of points (which are coplanar by construction) is critically flawed from the beginning. Indeed, various attributes of point triplets have been proposed for point pattern recognition—but many of these quantities clearly change when the triplet is viewed from different vantage points. For example, attributes like triangle interior angles or ratios of pairwise distances are only invariant under the action of the 2D similarity group G =S(2) that describes 2D translation, rotation, reflection, and scaling—which is sometimes a good approximation for a narrow field-of-view (such that projection is approximately orthographic instead of perspective) and when the camera boresight is perpendicular to the plane of the point triad (e.g., nadar pointing camera).

Sets of approximately coplanar points occur often in practice. For example, during terrain relative navigation (TRN) in scenarios where the terrain relief is small compared to the image footprint (as often happens at high altitude). Another common example is vision-based rendezvous with artificial satellites, where it is common for human-built structures to have flat surfaces containing keypoints suitable for navigation.

Set of Coplanar Conics

To develop the invariants for a set of coplanar conics, suppose there exists a homogeneous point on the plane $\bar{\boldsymbol{x}} = [x, y, 1]$ lying on the conic locus. In this case, one may describe the conic locus by the 3×3 symmetric matrix $\boldsymbol{C}$ of arbitrary scale that satisfies the equation

$$\bar{\boldsymbol{x}}^T \boldsymbol{C} \bar{\boldsymbol{x}} = 0 \tag{28}$$

When viewed from an arbitrary vantage point, substitution of Eq. (25) shows that the action of PGL(3) on this conic locus is described by [5]

$$\boldsymbol{B} \propto \boldsymbol{H}^{-T} \boldsymbol{C} \boldsymbol{H}^{-1} \tag{29}$$

where $\boldsymbol{B}$ describes the apparent conic locus in the image and $\boldsymbol{H}$ is the homography from Eq. (24).

Therefore, suppose we have objects whose models take the form $\mathcal{M} = (\boldsymbol{C}_1, \boldsymbol{C}_2, \ldots, \boldsymbol{C}_d)$. Such a d-tuple of coplanar conics is known to possess $5d-8$ independent invariants for $d \geq 2$ conics in general position on the plane. The existence of invariants in this situation, as well as the most common derivation approach, has been known since at least the 1950s [29]. It received extensive study within the context of computer vision in the 1990s [27, 30, 31, 32], and has been more recently adapted to crater-based spacecraft navigation [5, 33, 34].

The invariants for a set (sometimes called a *net*) of d conics may be found by looking at the intersection points of the conics—noting that conics that do not physically intersect over the real numbers still intersect over the complex numbers. Details of the derivation are provided in Section 6.2 of Ref. [5], which is an extension of the approach put forward by Semple and Kneebone [29]. For a pair of conics, one finds the well-known and widely used result

$$I_{ij} = \text{Tr}\left[\boldsymbol{C}_i^{-1} \boldsymbol{C}_j\right] = \text{Tr}\left[\boldsymbol{B}_i^{-1} \boldsymbol{B}_j\right] \tag{30}$$

$$I_{ji} = \text{Tr}\left[\boldsymbol{C}_j^{-1} \boldsymbol{C}_i\right] = \text{Tr}\left[\boldsymbol{B}_j^{-1} \boldsymbol{B}_i\right] \tag{31}$$

Note here that the middle term is built directly from the model [e.g., $f(\mathcal{M})$] and right-hand term is built directly form the image measurements [e.g., $h(\mathcal{I})$] and that these are numerically equivalent for any homography $\boldsymbol{H}$ that maps $\boldsymbol{C}$ to $\boldsymbol{B}$. For a d-tuple of conics, we obtain $5d-8$ independent invariants, consisting of all possible conic pairwise invariants from above—plus new invariants involving larger combinations of conics. For example, a crater triplet has 7 invariants: $I_{ij}, I_{ji}, I_{ik}, I_{ki}, I_{jk}, I_{kj}, I_{ijk}$. The new invariant I_{ijk} is [5]

$$I_{ijk} = \text{Tr}\left\{\left[(\boldsymbol{C}_j + \boldsymbol{C}_k)^* - (\boldsymbol{C}_j - \boldsymbol{C}_k)^*\right] \boldsymbol{C}_i\right\} = \text{Tr}\left\{\left[(\boldsymbol{B}_j + \boldsymbol{B}_k)^* - (\boldsymbol{B}_j - \boldsymbol{B}_k)^*\right] \boldsymbol{B}_i\right\} \tag{32}$$

where $\boldsymbol{C}^*$ is the adjugate matrix of $\boldsymbol{C}$. The procedure of Ref. [5] may be followed for any number of conics, though there does not seem to be much practical need for more than three. Nevertheless, invaraints for larger sets of conics are discussed in Ref. [31].

Since impact craters on planetary bodies are known to elliptical in shape [5], a set of coplanar conics is a good approximation for patterns of craters that are close to one another (i.e., where the curvature of the celestial body is small compared to the extent of the crater pattern).

Combinations of Coplanar Points, Lines, and Conics

Invariants also exist for various combinations of coplanar points, lines, and conics under the action of PGL(3). A few of the most common examples are discussed in Ref [27]. Of particular note is the invariant for a conic and two lines, which most often takes the form of a Cayley-Klein metric (and is of importance for the recognition of non-coplanar conics [5]). The existence of view invariants for mixed geometric shapes provides some flexibility in applying invariant theory to the varied situations that arise in practice.

Star Trackers

Stars are very far away and so they usually appear as unresolved objects (point sources) in digital images. Moreover, because the distances are so great, it is often possible to ignore parallax and so the star directions at any given epoch are fixed points on the celestial sphere—and are therefore fixed points in $\mathbb{P}^2$. Since the stars' apparent location in an image are also points in $\mathbb{P}^2$, we are interested in the mapping $\pi : \mathbb{P}^2 \to \mathbb{P}^2$.

Thus, when the star tracker is uncalibrated (or very poorly calibrated), the projection of stars into the image is described by PGL(3) and the problem is identical to point matching for a planar scene. This observation was made in Ref. [4] and we note that an asterism for a generic uncalibrated camera has $2d - 8$ invariants.

In the case of star trackers, various assumptions about the camera lead to different group actions and allow for the construction of different invaraints. There are three special cases to consider beyond the generic case mentioned above, and these special cases are sometimes very helpful when trying to recognize star patterns—especially when solving the so-called ”lost-in-space” star identification problem (which is sometimes called “blind astrometric calibration”). Each of these cases are now briefly discussed and a more detailed treatment may be found in Ref. [4].

First, in the case of a well-calibrated camera (i.e., when the camera calibraiton matrix $\boldsymbol{K}$ is known), points on the image plane may be unambiguously converted into directions in the sensor frame. Thus, the star directions from the catalog are related to the observed star directions by the action of $\mathrm{SO}(3)$. In this case, there are $2d-3$ invariants. The most common invariant is inter-star angle, but it is straightforward to conceive of other invaraints that are algebraic functions of the inter-star angles (e.g., dihedral angle).

Second, in the case that a well-calibrated camera is also known to have a very small FOV, the projection is nearly orthographic and catalog star patterns are related to observed star patterns by the action of $\mathrm{E}(2)$. In this case there are also $2d - 3$ invariants, so the narrow FOV assumption offers little practical advantage over the wide FOV case. Thus, it is usually recommended to avoid this simplification.

Third, in the case of an uncalibrated camera with a narrow FOV, we may approximate the catalog-to-image mapping by the action of S(2). This situation is commonly encountered with telescopes (rather than cameras) and forms the unstated underpinning of popular pipelines like Astrometry.net [3].

Nearly every class of existing star identification algorithm falls into one of the above four categories, with invariants based on $\mathrm{PGL}(3)$, $\mathrm{SO}(3)$, $\mathrm{E}(2)$, or $\mathrm{S}(2)$ [4]. A rereading of the star identification literature [1, 2] with this in mind will reveal that many purportedly different algorithms are actually geometrically equivalent—and differ mainly in the choice of data structure used to query the catalog. Moreover, since efficient means of performing nearest neighbor queries and range queries are now quite mature [35, 36, 37], it is often advantageous to use these algorithms rather than some of the ad hoc techniques existing only in the star identification literature. This is especially true as the size of the star catalog becomes very large (e.g., using the Gaia database [38, 39]).

3D Scenes

The most challenging of the scenarios considered in this work are view invariants that can be constructed from camera images of truly 3D objects. Indeed, we find these to not always exist—which is often a very inconvenient truth that places substantial limitation on what is achievable with images alone. Despite this, when view invariants do exist, we find them to be a powerful tool for object recognition.

Set of 3D Points in General Position

Suppose we have objects whose models take the form $\mathcal{M} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_d)$, where the points $\{\boldsymbol{p}_i\}_{i=1}^d \in \mathbb{R}^3$ are in general position. It has been known for some time that, if the camera is also in general position, then view invariants do not exist [21, 22, 23]. This is true even if the camera orientation is known (as discussed in one the earlier examples, and as implied by the discussion of [23]). However, if we constrain the camera to lie along a known line, a d-tuple of points has d invariants corresponding to the orientation of the plane formed by camera's path and the observed point. The existence of such invariants were discussed in Ref. [23] and it was proven in Ref. [24] that a linear path is the only camera path along which invariants exist for a d-tuple of arbitrarily placed points.

Set of Non-Coplanar Conics on a Quadric Surface

Since celestial bodies are not flat plates, the conics describing crater patterns are only nearly planar if the craters are very close to one another. When looking at large patterns of craters, the global shape of the celestial body must be considered. It was shown in Ref. [5] that no view invariants exist for a d-tuple of 3D conics in general position. Fortunately, since large celestial bodies have an ellipsoidal shape due to selfgravitation and rotational dynamics [40], we have a much more structured problem. Indeed, it was proven in Ref. [5] that constraining a d-tuple of conics to lie on the surface of a non-degenerate quadric (e.g., an ellipsoid) introduces invariants. Specifically, a d-tuple of conics lying on a non-degenerate quadric possess $3d - 6$ independent invariants which ultimately take the form of a Cayley-Klein metric. The development is somewhat involved, though the final implementation is simple to code. Specifically, if $\boldsymbol{A}_i, \boldsymbol{A}_j, \boldsymbol{A}_k$ describe the conic locus of three apparent crater rims in an image, then one of the three $(3 \times 3 - 6 = 3)$ view invariants is

$$J_i = \text{acosh} \left\{ \frac{\| \boldsymbol{\ell}_{ij}^T \boldsymbol{A}_i^* \boldsymbol{\ell}_{ik} \|}{\sqrt{\left(\boldsymbol{\ell}_{ij}^T \boldsymbol{A}_i^* \boldsymbol{\ell}_{ij} \right) \left(\boldsymbol{\ell}_{ik}^T \boldsymbol{A}_i^* \boldsymbol{\ell}_{ik} \right)}} \right\} \tag{33}$$

where $\boldsymbol{\ell}_{ij}$ is one of the lines joining two of the (possibly complex valued) intersection points of the image conics $\boldsymbol{A}_i$ and $\boldsymbol{A}_j$. Details of finding $\boldsymbol{\ell}_{ij}$ and $\boldsymbol{\ell}_{ik}$ are found in Ref. [5]. Two more independent invariants (J_j and J_k) may be computed in a similar fashion. The invariants J_i, J_j, J_k will remain the same for this triplet of elliptical craters, regardless of the vantage point (i.e. pose) from which the pinhole camera views the crater pattern. Thus, these invariants to be used to easily recognize a pattern of craters for orbital TRN, where the curvature of the observed body is important. Some practical details about crater catalog curation and structuring are also discussed in Ref. [5].

CONCLUSIONS

It is often necessary to associate sensor measurements with a corresponding model for the purposes of spacecraft navigation, orbit determination, and other space exploration applications. This can be a challenging task, especially when the pose (i.e., relative position and attitude) between the sensor and observed object is unknown. One algorithmically efficient and theoretically rigorous means of accomplishing this task is with invariants. Therefore, this work reviews some of the key aspects of invariant theory and how it may be used within the context of spacecraft navigation and orbit determination. In particular, concepts related to invariants and view invariants (and their types: e.g., polynomial, rational) are developed. Techniques for enumerating the number of independent invariants are also presented. Finally, a brief summary of the most important known invariants is presented—along with an extensive set of references to more detailed developments.

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