Leveraging Dynamical System Theory to Incorporate Design Constraints in Multidisciplinary Design

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This work views the multidisciplinary design problem as a discrete dynamical system. The dynamical system is a result of the convergence of the design, which requires that each of the disciplines use the same information, have a common set of assumptions, and satisfy the constraints imposed on the design. An additional dynamical system is formed during the optimization of the design. Using optimal control theory for dynamical systems, allows design constraints to be easily imposed on the design variables and the outputs of each of the contributing analyses at the same level in the design hierarchy. By coordinating these constraints at the same level in the design hierarchy, parts of the design space that have conflicting constraints between design variables and contributing analyses can be avoided. Finally, by viewing the constraints using control theory, established, efficient solution procedures can be invoked in the design process.

Nomenclature

$(\cdot)^T$	Matrix transpose of $(\cdot)$
$(\cdot)^{-1}$	Matrix inverse of $(\cdot)$
$\boldsymbol{\lambda}$	Lagrange multiplier
$\boldsymbol{\mu}$	Lagrange multiplier
$\boldsymbol{\nu}$	Lagrange multiplier
$\mathbb{R}$	Set of real numbers
$\mathbf{0}_q$	$q \times 1$ vector of zeros
$\mathbf{u}$	Control variable in the optimal control problem
$\mathcal{L}(\cdot)$	Optimal control path cost
$\mathcal{P}^{(i)}$	Convex cone for the discrete optimal control problem
$\mathcal{Q}^{(i)}$	Convex cone for the discrete optimal control problem
$\mathcal{U}$	Set of admissible inputs (controls)
$\phi(\cdot)$	Terminal state cost
Σ	Set of admissible states
$H(\cdot)$	Hamiltonian
$L(\cdot)$	Lagrangian

I. Overview

In Refs. 1 and 2, a methodology that rapidly obtains the mean and a bound on the variance of a multidisciplinary design was developed. This new methodology is possible by viewing the multidisciplinary design problem as a dynamical system. In fact, two discrete dynamical systems are formed, both of which are the resultant of a root-finding problem. The first dynamical system addresses the identification of feasible designs by ensuring each of the contributing analyses (CAs) use the same output. For this system, the state is the contributing analyses (CAs) output. The second dynamical system formed is that of finding an optimum of the design, where the state is the objective function of the optimization.

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The concepts described subsequently address the handling of constraints that are a function of both the CA output and the design variables when viewing the design problem as a dynamical system. This is done by approaching the problem as an optimal control problem which allows equality and inequality constraints to be placed on the state (the CA output) and control (the design variables) of the system. This approach enables the handling of both the design variable and CA constraints is done at the same level of the design hierarchy, which provides coordination in the search of feasible and optimal solutions.

II. Discrete Dynamical Systems

The techniques developed in Ref. 1 rely on discrete dynamical systems. That is, a dynamical system of the form

$$\left.\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k, k) \\ \mathbf{y}_k &= \mathbf{g}(\mathbf{x}_k, \mathbf{u}_k, k) \end{aligned}\right\} \tag{1}$$

where $\mathbf{x}$ is the state of the system, $\mathbf{f}$ is a function which describes the time evolution of the system, $\mathbf{u}$ is the input into the system, and k is the iterate number. A specific instance of Eq. (1) that is used in Ref. 1 is a linear, discrete dynamical system, which is given by

$$\left.\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}_k\mathbf{x}_k + \mathbf{B}_k\mathbf{u}_k \\ \mathbf{y}_k &= \mathbf{C}_k\mathbf{x}_k + \mathbf{D}_k\mathbf{u}_k \end{aligned}\right\} \tag{2}$$

Graphically, this linear discrete dynamical system is shown in the block diagram shown in Fig. 1.

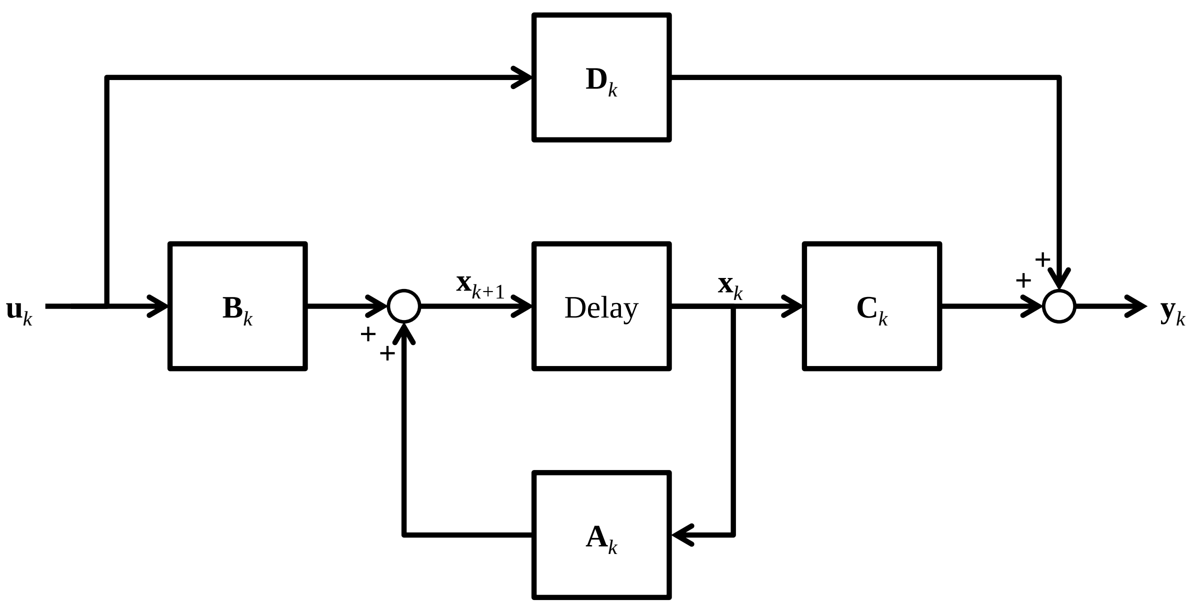


Figure 1. Block diagram of a linear, discrete dynamical system.

III. The Multidisciplinary Design Problem as a Dynamical System

III..1. Identification of Feasible Designs

Identifying feasible designs in multidisciplinary systems can be thought of as the process of finding the root of a function. Consider a multidisciplinary problem where the analysis variables are described by a multivariable function $\mathbf{f}(\mathbf{u}, \mathbf{p})$ where $\mathbf{u}$ are the design variables and $\mathbf{p}$ are the parameters of the problem. Assume that the requirements of the design are given by only equality constraints that are a function of the performance of the system. The performance of the design is described by a multi-variable mapping $\mathbf{g}(\mathbf{f}(\mathbf{u}, \mathbf{p}))$ and the requirements are given by $\mathbf{z}$. In order to meet the requirements it is necessary to adjust the design variables $\mathbf{u}$ so that

$$\mathbf{z} = \mathbf{g}(\mathbf{f}(\mathbf{u}, \mathbf{p})) \tag{3}$$

The solution $\mathbf{u}^*$ of Eq. (3) is the root of the system. Since identifying feasible designs within the multidisciplinary design problem requires finding the value of $\mathbf{u}$ that satisfies Eq. (3), this process can be thought of as a root-finding problem when an iterative solution method is chosen.

Many numerical methods for finding the root of a function, $\mathbf{g}(\mathbf{u})$, are dynamical systems since they rely on iterative schemes to identify the root.[3] For instance, the bisection method, secant method, function iteration method, and Newton's method are all iterative techniques that satisfy the requirements of a dynamical system.

III..2. Design Optimization

In order for a converged design to be an optimum with respect to some objective function, its performance needs to be evaluated with respect to other potential designs. The general optimization problem is formulated as

$$\left.\begin{array}{lll}\textbf{Minimize:} & \mathcal{J}(\mathbf{u}, \mathbf{p}) & \\ \textbf{Subject to:} & \mathbf{g}_i(\mathbf{u}, \mathbf{p}) \leq \mathbf{0}, & i = 1, \ldots, n_g \\ & \mathbf{h}_j(\mathbf{u}, \mathbf{p}) = \mathbf{0}, & j = 1, \ldots, n_h \\ \textbf{By varying:} & \mathbf{u} & \end{array}\right\} \tag{4}$$

The solution requires an auxiliary function, the Lagrangian, to be defined as

$$L(\mathbf{u}, \mathbf{p}, \boldsymbol{\lambda}) = \mathcal{J}(\mathbf{u}, \mathbf{p}) + \sum_{i=1}^{n_g} \lambda_i \mathbf{g}_i(\mathbf{u}, \mathbf{p}) + \sum_{j=1}^{n_h} \lambda_{n_g+j} \mathbf{h}_j(\mathbf{u}, \mathbf{p}) \tag{5}$$

to be found. The first-order, necessary conditions for $\mathbf{u}^*$ to be an optimum are[4]

1. $\mathbf{u}^*$ is feasible
2. $\lambda_i \mathbf{g}_i(\mathbf{u}^*) = \mathbf{0} \quad i = 1, \ldots, n_g$ and $\lambda_i \geq 0$
3. $\nabla L(\mathbf{u}, \mathbf{p}, \lambda) = \nabla \mathcal{J}(\mathbf{u}, \mathbf{p}) + \sum_{i=1}^{n_g} \lambda_i \nabla \mathbf{g}_i(\mathbf{u}, \mathbf{p}) + \sum_{j=1}^{n_h} \lambda_{n_g+j} \nabla \mathbf{h}_j(\mathbf{u}, \mathbf{p}) = \mathbf{0}$ with all $\lambda_i \geq 0$ and λ_{n_g+j} unrestricted in sign

Each of the necessary conditions is a root-finding problem by itself. As previously discussed, the process of finding $\mathbf{u}^*$ is a process of root-finding. Additionally, as apparent from the form of the relationships, the other two necessary conditions for optimality may also be obtained through a root-finding technique. The values of the Lagrange multipliers for the inequality constraints, $\hat{\boldsymbol{\lambda}} = \left(\lambda_1 \ldots \lambda_{n_g}\right)^T$ can be found from the relation

$$\hat{\boldsymbol{\lambda}}(\mathbf{x}^*) = -[\mathbf{A}^T \mathbf{A}]^{-1} \mathbf{A}^T \nabla \mathcal{J}(\mathbf{x}^*) \tag{6}$$

where

$$\mathbf{A}(\mathbf{x}^*) = \begin{pmatrix} \nabla \mathbf{g}_1(\mathbf{x}^*) & \nabla \mathbf{g}_2(\mathbf{x}^*) & \ldots & \nabla \mathbf{g}_{n_g}(\mathbf{x}^*) \end{pmatrix} \tag{7}$$

III..3. Identifying an Optimal Multidisciplinary Design

Multidisciplinary design optimization can be broken down into two steps: (1) identifying feasible designs and (2) identifying the optimal design from the set of feasible candidates. As discussed, both of these steps are root-finding problems. With the choice of an appropriate iterative numerical root-finding scheme, each of these individual steps can be posed as dynamical systems. When combined together, a nested root-finding problem results.

III.A. Identifying an Optimal Multidisciplinary Design

Multidisciplinary design optimization can be broken down into two steps: (1) identifying feasible designs and (2) identifying the optimal design from the set of feasible candidates. As discussed, both of these steps are root-finding problems. With the choice of an appropriate iterative numerical root-finding scheme, each of these individual steps can be posed as dynamical systems. When combined together, a nested root-finding problem results, whereby the function being optimized is actually a root-finding problem itself.

IV. Constraints Using Optimal Control Theory

As an alternative to the general design solution procedure, consider an alternative, optimal control. Again, the solution procedure is a root-finding problem, however, optimal control techniques adjoin a "tangency" condition for constraints that are a function of the input or state. This allows a more general framework to handle various constraints that may vary depending on the current input values as well as a direct way to handle constraints that are a function of the design variables and the CA values.

IV.A. Continuous Dynamical Systems

To first illustrate the handling of constraints using optimal control theory, consider the general continuous-time optimal control problem given by

$$\left.\begin{array}{ll}\textbf{Minimize}: & \mathcal{J} = \phi(\mathbf{x}(t_f), t_f) + \displaystyle\int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}(t), \mathbf{u}(t), t)dt \\ \textbf{Subject to}: & \dfrac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \\ & \mathbf{u}(t) \in \mathcal{U} \\ & \mathbf{x}(t) \in \Sigma \\ \textbf{By varying}: & \mathbf{u}(t)\end{array}\right\} \tag{8}$$

In Eq. (8), ϕ is the terminal state cost, $\mathcal{L}$ is the transient or path cost, $\mathcal{U}$ is the set of admissible controls, and Σ is the set of admissible states. Suppose that there is a constraint on the state given by

$$\mathbf{S}(\mathbf{x}, t) = 0 \tag{9}$$

Differentiating Eq. (9) with respect to time, one obtains

$$\dot{\mathbf{S}} = \frac{\partial \mathbf{S}}{\partial t} + \frac{\partial \mathbf{S}}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt} = \mathbf{0} \tag{10}$$

Substituting the state equation into this result yields

$$\dot{\mathbf{S}} = \frac{\partial \mathbf{S}}{\partial t} + \frac{\partial \mathbf{S}}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) = \mathbf{0} \tag{11}$$

and allows for a technique to yield the optimal control, $\mathbf{u}(t)$, that minimizes $\mathcal{J}$ and meets the equality constraint on the state.[5,6] If the control is not explicit in Eq. (10), then the process of differentiating $\mathbf{S}$ and substituting the state equation is continued until the control is explicit in the equation to form a set of q point relationships $\{\mathbf{S}^{(n)}\}, \quad n = 0, \ldots, q-1$, where n is the order of the derivative. These tangency conditions can be adjoined using Lagrange multipliers to the path cost, $\mathcal{L}$, to solve for the optimal control history.

Inequality constraints of the form

$$\bar{\mathbf{S}}(\mathbf{x}, t) \leq \mathbf{0} \tag{12}$$

can be handled similarly.[5,6] In this case, the solution process depends on whether or not the state is on the boundary. If it is on the boundary, the same solution process to equality constraints is followed, while for off-boundary solutions, the terms are ignored. This results in a multiple sub-arc solution, although fundamentally the process is identical to the equality constraint case.

IV.B. Discrete Dynamical Systems

For the discrete optimal control problem posed as[7]

$$\left.\begin{array}{ll}\textbf{Minimize:} & \mathcal{J}(\mathbf{x},\mathbf{u}) = \displaystyle\sum_{k=0}^{n-1} \mathcal{L}(\mathbf{x}_k, \mathbf{u}_{k+1}, k) \\ \textbf{Subject to:} & \mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_{k+1}, k), \quad \forall k \in \{0,\dots,n-1\} \\ & \mathbf{u}_k \in \mathcal{U}(\mathbf{x}_{k-1}), \quad \forall k \in \{1,\dots,n\} \\ & \mathbf{x}_k \in \Sigma, \quad \forall k \in \{0,\dots,n-1\} \\ \textbf{By varying:} & \mathbf{u}_k, \quad \forall k \in \{1,\dots,n\}\end{array}\right\} \tag{13}$$

the optimal control satisfies the following necessary conditions which allows it to be satisfied. First, assume the following regarding the analysis domain:

1. $\mathcal{U}$ is defined for each $\mathbf{x} \in \Sigma$ by equality and inequality constraints of the form

$$\mathbf{g}(\mathbf{x},\mathbf{u}) \leq \mathbf{0}; \quad \mathbf{g} \in \mathbb{R}^{q_t \times 1} \tag{14}$$

$$\mathbf{h}(\mathbf{x},\mathbf{u}) = \mathbf{0}; \quad \mathbf{h} \in \mathbb{R}^{k_t \times 1} \tag{15}$$

2. Σ is defined by the equality and inequality constraints of the form

$$\mathbf{w}(\mathbf{x}) = \mathbf{0}; \quad \mathbf{w} \in \mathbb{R}^{r_t \times 1} \tag{16}$$

$$\boldsymbol{\omega}(\mathbf{x}) \leq \mathbf{0}; \quad \boldsymbol{\omega} \in \mathbb{R}^{p_t \times 1} \tag{17}$$

3. There exists convex cones, $\mathcal{P}^{(i)}$ and $\mathcal{Q}^{(i)}$ with vertices at $\mathbf{x}(\tau)$ that cover Σ for all $\tau = 0, 1, \dots, n$
4. There exists a scalar $\psi_0 \leq 0$ and vectors

$$\boldsymbol{\psi}_t = [\psi_{1,t}, \dots, \psi_{n,t}]^T, \quad \forall t \in \{1,\dots,n\} \tag{18}$$

$$\boldsymbol{\gamma}_t = [\gamma_{1,t}, \dots, \gamma_{k_t,t}]^T, \quad \forall t \in \{1,\dots,n\} \tag{19}$$

$$\boldsymbol{\lambda}_t = [\lambda_{1,t}, \dots, \lambda_{q_t,t}]^T, \quad \forall t \in \{1,\dots,n\} \tag{20}$$

$$\boldsymbol{\mu}_t = [\mu_{1,t}, \dots, \mu_{r_t,t}]^T, \quad \forall t \in \{0,\dots,n\} \tag{21}$$

$$\boldsymbol{\nu}_t = [\nu_{1,t}, \dots, \nu_{p_t,t}]^T, \quad \forall t \in \{0,\dots,n\} \tag{22}$$

$$\mathbf{b}^{(i)}, \quad i = 1, \dots, n \quad \forall t \in \{0,\dots,n\} \tag{23}$$

   such that the direction of $\mathbf{b}^{(i)}$ lies in the dual cone $\mathcal{D}(\mathcal{P}^{(i)})$

5. Let a scalar function $H_t(\mathbf{x},\mathbf{u})$ be defined as

$$H_t(\mathbf{x},\mathbf{u}) = \psi_0 \mathcal{L}(\mathbf{x},\mathbf{u},t) + \boldsymbol{\psi}_t^T \mathbf{f}(\mathbf{x},\mathbf{u},t) + \boldsymbol{\gamma}_t^T \mathbf{h}(\mathbf{x},\mathbf{u}) + \boldsymbol{\lambda}_t^T \mathbf{g}(\mathbf{x},\mathbf{u}) \tag{24}$$

The necessary conditions for an optimal control to exist are then given by:

1. If $\psi_0 = 0$ then for at least one t at least one of the vectors $\boldsymbol{\psi}_t$, $\boldsymbol{\gamma}_t$, $\boldsymbol{\lambda}_t$, $\boldsymbol{\mu}_t$, $\boldsymbol{\nu}_t$, or $\mathbf{b}^{(i)}$ is non-zero
2. For all $t = 0, \dots, n$ and any vector $\boldsymbol{\delta}\mathbf{x}$ whose direction lies in the intersection of the cones $\mathcal{Q}^{(i)}$, the following inequality holds

$$\left(-\boldsymbol{\psi}_t + \frac{\partial H_{t+1}(\mathbf{x},\mathbf{u})}{\partial \mathbf{x}} + \boldsymbol{\mu}_t^T \frac{\partial \mathbf{w}(\mathbf{x})}{\partial \mathbf{x}} + \boldsymbol{\nu}_t^T \frac{\partial \boldsymbol{\omega}(\mathbf{x})}{\partial \mathbf{x}} - \sum_{i=1}^{n_t} \mathbf{b}^{(i)} \right)^T \boldsymbol{\delta}\mathbf{x} \leq 0$$

   where it is assumed that $\boldsymbol{\psi}_0 = \mathbf{0}$ and $H_{n+1} = 0$

3. $\dfrac{\partial H(\mathbf{x}_{t-1}, \mathbf{u}_t)}{\partial \mathbf{u}} = \mathbf{0}, \quad \forall t \in \{1,\dots,n\}$
4. $\lambda_{\alpha,t} \leq 0$, $\lambda_{\alpha,t} g_t^\alpha(\mathbf{x}_{t-1}, \mathbf{u}_t) = 0$ $\forall \alpha \in \{1 \dots, q_t\}$ and $t \in \{1, \dots, n\}$
5. $\nu_{\alpha,t} \leq 0$, $\nu_{\alpha,t} \omega_t^\alpha(\mathbf{x}(t)) = 0$ $\forall \alpha \in \{1, \dots, p_t\}$ and $t \in \{0, \dots, n\}$

The proof of these conditions minimizing $\mathcal{J}(\mathbf{x},\mathbf{u})$ is found in Ref. 7. Note that the process of adjoining the tangents of the state and control constraints to the objective functional in the second criteria is nearly identical to that of the continuous case with the additional requirement that the space is convex (as defined by the intersection of the cones).

IV.C. Solution Methods

The discrete problem as posed is a nonlinear programming (NLP) problem, which has many known solution technique which include gradient methods, quadratic programming, sequential quadratic programming, and interior-point methods.[4,7,8]

Direct methods to the continuous problem also approach the solution procedure to the optimal control problem as a discretized problem. This again gives rise to an NLP. However, unlike the discrete formulation described previously, the direct solution to the continuous problem may require the use of penalty functions for constraints. Alternatively, indirect methods would approximate the discrete problem as a continuous problem (*i.e.*, taking the step size caused by the iteration to zero) and then solves a boundary value problem. This allows the constraints to be handled directly using Lagrange multipliers. The validity of viewing the discrete problem as a continuous problem has been shown to work well for multidisciplinary design problems, as the solution set is in general more restrictive (including continuity requirements on the constraints) than the discrete problem; however, solutions may not always exist.

A comparison of the different solution methods is shown in Table 1.

Table 1. Comparison of Solution Techniques.

	Advantages	**Disadvantages**
Direct Methods	Large Region of Attraction Large Number of NLP Solvers	Computationally Intensive Convexity Requirement Use of Penalty Functions
Indirect Methods	Fast Convergence Solution Optimality Exact Solution of Constraints	Small Region of Attraction Solutions May Not Exist

IV.D. Solution Search Coordination

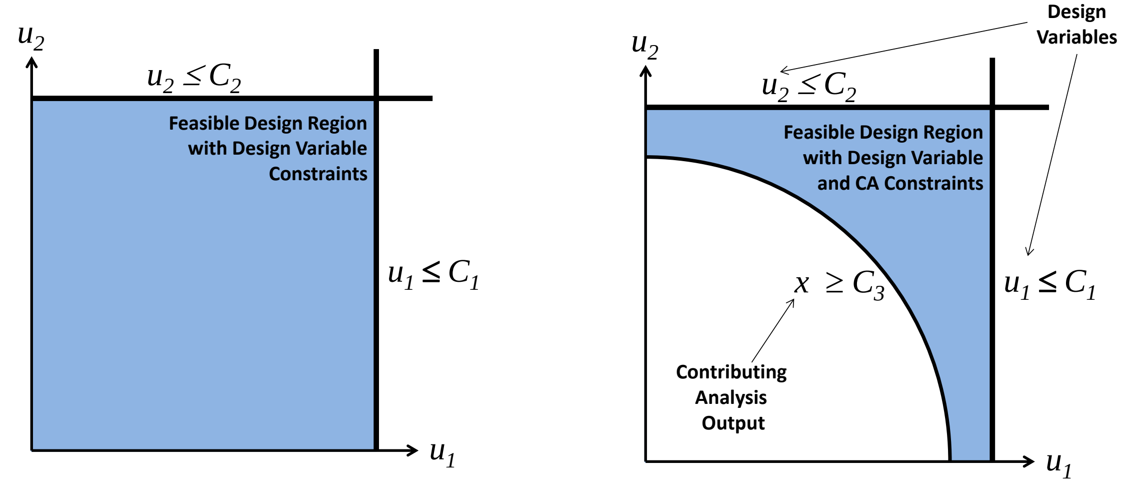
It has already been mentioned that approaching the constraint problem in the multidisciplinary design problem from the optimal control perspective enables the *explicit* handling of constraints that are a function of both the CA output (*i.e.*, the state, $\mathbf{x}$) and the design variables (*i.e.*, the control, $\mathbf{u}$). This is opposed to the standard design optimization problem given in Eq. (4) where the inequality and equality constraints, $\mathbf{g}_i(\mathbf{u}, \mathbf{p})$ and $\mathbf{h}_i(\mathbf{u}, \mathbf{p})$, are explicit functions of the design variables only. To accommodate CA constraints, one would have to apply penalty function (or similar) technique to the problem which means that the enforcement of the constraints may not be exact.

By accommodating both design variables and CA constraints at the same level in the optimization hierarchy, a reduced design space can be searched which eliminates the design region with conflicting constraints. This is shown schematically in Fig. 2.

In Fig. 2(a), the design region resulting from traditional optimization where constraints are only a function of design variables is shown which implies that there is a relatively large feasible region. While in Fig. 2(b) the feasible region that actually exists is shown as it accounts for all possible constraints in the problem. The results seen in Fig. 2(b) is the region that is searched by approaching this problem from the optimal control perspective.

IV.E. Use of Continuation Methods

It should also be mentioned that using the optimal control approach to approach an optimal design allows a continuation-type approach to be applied to the design variables to enable convergence for potentially difficult to converge designs. This would constrain the design variables to have initially small variations around a solution that is known to be valid and then grow as a function of the iteration until the entire design variable range can be explored.



(a) (b)

Figure 2. Feasible design space accounting for (a) design variable constraints only and (b) design variable and contributing analysis constraints.

V. Design Examples

The following examples employ a rapid robust design methodology as described in Ref. 1 to demonstrate using optimal control theory to impose design variable and CA output constraints. The nomenclature in the section that follows is consistent with that work.

V.A. Linear, Three Contributing Analysis System

Consider the linear, three CA system shown in Fig. 3 where each CA is scalar. In this case, it is desired to

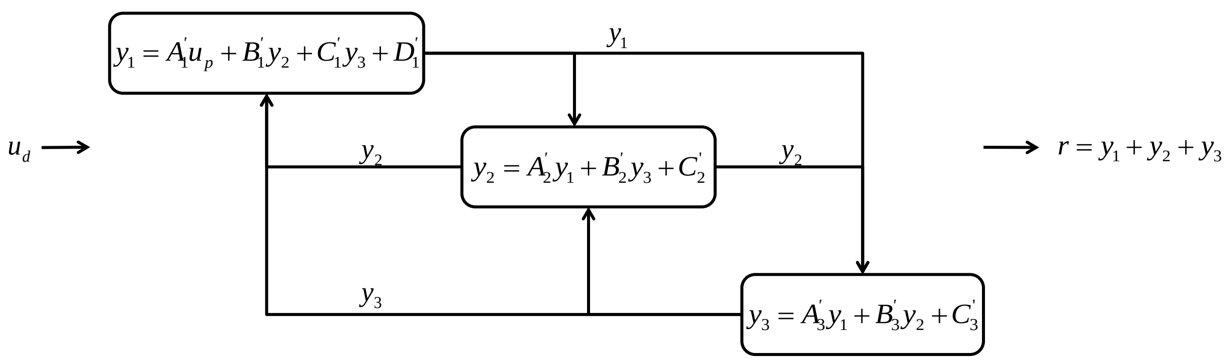


Figure 3. Three contributing analysis multidisciplinary design.

find $u_d \in \mathbb{R}$ that minimizes the summation of the CAs output while being within the unit cube centered at the origin. In other words

$$\left.\begin{array}{ll}\textbf{Minimize:} & \mathcal{J} = y_1 + y_2 + y_3 \\ \textbf{Subject to:} & y_1, y_2, y_3 \in [-1, 1] \\ \textbf{By varying:} & u_d\end{array}\right\}$$

The problem as given has already been decomposed into the representative contributing analyses. For the first CA, $\mathbf{y_1}$, the functional form of the CA is as follows

$$y_1 = \begin{pmatrix} 0 & B_1' & C_1' \end{pmatrix} \mathbf{y} + \begin{pmatrix} A_1' \end{pmatrix} u_d + \begin{pmatrix} \mathbf{0} \end{pmatrix} \mathbf{u_p} + D_1'$$

Similarly, for the second CA, the functional form is given by

$$y_2 = \begin{pmatrix} A_2' & 0 & B_2' \end{pmatrix} \mathbf{y} + \begin{pmatrix} 0 \end{pmatrix} u_d + \begin{pmatrix} \mathbf{0} \end{pmatrix} \mathbf{u_p} + C_2'$$

and the third CA

$$y_3 = \begin{pmatrix} A_3' & B_3' & 0 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 0 \end{pmatrix} u_d + \begin{pmatrix} \mathbf{0} \end{pmatrix} \mathbf{u_p} + C_3'$$

Hence,

$$\begin{aligned} \mathbf{A_1} &= \begin{pmatrix} 0 & B_1' & C_1' \end{pmatrix} & \mathbf{B_1} &= \begin{pmatrix} A_1' \end{pmatrix} & \mathbf{C_1} &= \begin{pmatrix} \mathbf{0} \end{pmatrix} & \mathbf{d_1} &= D_1' \\ \mathbf{A_2} &= \begin{pmatrix} A_2' & 0 & B_2' \end{pmatrix} & \mathbf{B_2} &= \begin{pmatrix} 0 \end{pmatrix} & \mathbf{C_2} &= \begin{pmatrix} \mathbf{0} \end{pmatrix} & \mathbf{d_2} &= C_2' \\ \mathbf{A_3} &= \begin{pmatrix} A_3' & B_3' & 0 \end{pmatrix} & \mathbf{B_3} &= \begin{pmatrix} 0 \end{pmatrix} & \mathbf{C_3} &= \begin{pmatrix} \mathbf{0} \end{pmatrix} & \mathbf{d_3} &= C_3' \end{aligned}$$

For this example, the fixed-point iteration equations are

$$\mathbf{\Lambda} = \begin{pmatrix} \mathbf{A_1} \\ \mathbf{A_2} \\ \mathbf{A_3} \end{pmatrix} = \begin{pmatrix} 0 & B_1' & C_1' \\ A_2' & 0 & B_2' \\ A_3' & B_3' & 0 \end{pmatrix}$$

$$\boldsymbol{\beta} = \begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \\ \mathbf{B_3} \end{pmatrix} = \begin{pmatrix} A_1' \\ 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{\gamma} = \begin{pmatrix} \mathbf{C_1} \\ \mathbf{C_2} \\ \mathbf{C_3} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\delta} = \begin{pmatrix} \mathbf{d_1} \\ \mathbf{d_2} \\ \mathbf{d_2} \end{pmatrix} = \begin{pmatrix} D_1' \\ C_2' \\ C_3' \end{pmatrix}$$

In the optimal control problem, the objective function is given by

$$\mathcal{J} = \mathbf{1}_3^T \mathbf{y}$$

The design space and constraints in this problem are inherently convex therefore the constraints can be directly formulated as

$$\mathbf{g}(\mathbf{y}, \mathbf{u}) = \begin{pmatrix} y_1 - 1 \\ y_2 - 1 \\ y_3 - 1 \\ -y_1 - 1 \\ -y_2 - 1 \\ -y_3 - 1 \end{pmatrix} = \begin{pmatrix} \mathbf{I_{3\times3}} \\ -\mathbf{I_{3\times3}} \end{pmatrix} \mathbf{y} - \mathbf{1_6} \leq \mathbf{0}_6$$

Additionally, there is an equality constraint for the control that states

$$\mathbf{h}(\mathbf{y}, \mathbf{u}) = \mathbf{u}_k - \mathbf{u}_{k-1} = \mathbf{0}, \quad \forall k \in 1, \ldots, n$$

Therefore, the Hamiltonian in the discrete optimal control problem is given by

$$H_t(\mathbf{y}, \mathbf{u}) = \psi_0 \mathbf{y} + \boldsymbol{\gamma}^T \left(\begin{pmatrix} \mathbf{I_{3\times3}} \\ -\mathbf{I_{3\times3}} \end{pmatrix} \mathbf{y} - \mathbf{1_6} \right) + \boldsymbol{\lambda}^T \left(\mathbf{u}_k - \mathbf{u}_{k-1} \right)$$

where the terms in this relation can be computed numerically. Alternatively, treating this as a continuous optimal control problem, the Hamiltonian would be given by

$$H(\mathbf{y}, \mathbf{u}) = \boldsymbol{\lambda}^T \left(\mathbf{\Lambda} \mathbf{y} + \boldsymbol{\beta} u_d - \dot{\mathbf{y}} \right) + \boldsymbol{\mu}^T \left(\begin{pmatrix} \mathbf{I_{3\times3}} \\ -\mathbf{I_{3\times3}} \end{pmatrix} \mathbf{y} - \mathbf{1_6} \right)$$

with the terminal constraint function given as

$$G(\mathbf{y}, \mathbf{u}) = y_1(t_f) + y_2(t_f) + y_3(t_f)$$

In both of these equations, the unknown terms can be computed numerically.

V.A.1. Design Results

The parameters used within the models are shown in Table 2.

Table 2. Parameters for the robust design of a three contributing analysis system.

Parameter	Value
$\boldsymbol{\Lambda}$	$\begin{pmatrix} 0 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 0 \end{pmatrix}$
$\boldsymbol{\beta}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
$\boldsymbol{\gamma}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
$\boldsymbol{\delta}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Approaching this as discrete optimal control problem, the optimal design is found using the Sparse Nonlinear Optimization (SNOPT), an NLP solver which uses a sequential quadratic programming algorithm.[9] This software package was also used for the discrete optimal control problem. Finally, a boundary value problem solver using collocation was used to solve the indirect problem. The solutions obtained for each of these techniques along with the number of function evaluations is shown in Table. 3.

Table 3. Design results for the Linear, Three Contributing Analysis System.

Method	u_d^*	$\mathcal{J}^*$	Function Evaluations
Discrete	-0.6665	2.499	21
Continuous, Direct	-0.6645	2.478	26
Continuous, Indirect	-0.6666	2.500	16

V.B. Robust Design of a Two Bar Truss

Consider the planar truss which consists of two elements with a vertical load at the mutual joint, as shown in Fig. 4 (adapted from Ref. 10).

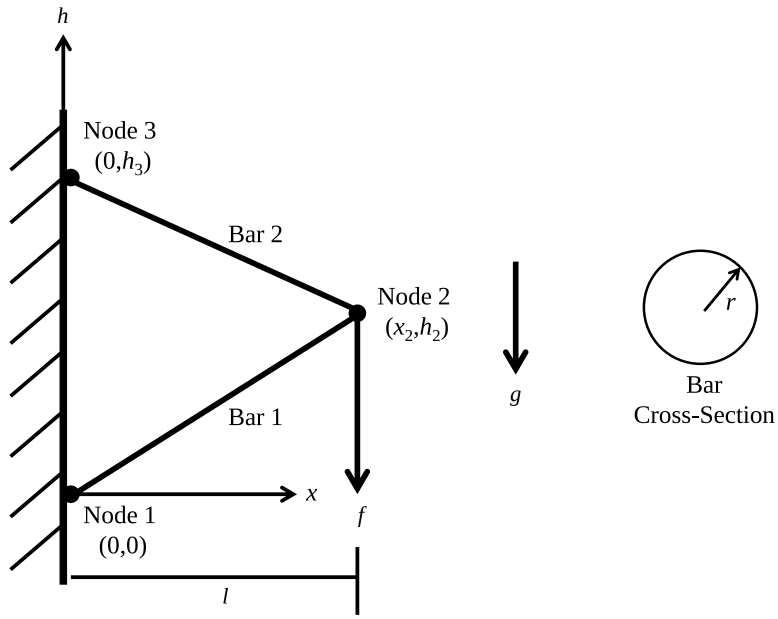


Figure 4. Two bar truss with a load at the mutual joint.

For this problem, it is desired to find the vertical position of nodes 2 and 3, h_2 and h_3, that minimize the weight of the truss while ensuring that the structure will not fail due to Euler buckling or yielding with some factor of safety given fixed values for the material properties, E, σ_y, and ρ, the load, f, and the bar geometry, r_1 and r_2. The horizontal position of node 2 is constrained to be l. In standard form, the deterministic problem is written as

$$\left.\begin{array}{ll}
\textbf{Minimize}: & \mathcal{J} = \rho\pi g\left(r_1^2 L_1 + r_2^2 L_2\right) = \rho\pi g\left(r_1^2\sqrt{l^2+h_2^2} + r_2^2\sqrt{l^2+(h_3-h_2)^2}\right) \\
\textbf{Subject to}: & g_1(h_2,h_3) = |T_1(h_2,h_3)| - \pi r_1^2\sigma_y \le 0 \\
 & g_2(h_2,h_3) = |T_2(h_2,h_3)| - \pi r_2^2\sigma_y \le 0 \\
 & g_3(h_2,h_3) = -T_1(h_2,h_3) - \dfrac{\pi^2 E I_1}{L_1^2} \le 0 \\
 & g_4(h_2,h_3) = -T_2(h_2,h_3) - \dfrac{\pi^2 E I_2}{L_2^2} \le 0 \\
\textbf{By varying}: & h_2, h_3
\end{array}\right\}$$

where L_1 and L_2 are the lengths of the two bars, respectively, I_1 and I_2 are the moments of inertia of the two bars ($I_i = \dfrac{1}{4}mr_i^2$), and $T_1(h_2,h_3)$ and $T_2(h_2,h_3)$ are the tensions in the two bars. In this formulation both the objective function and constraints are nonlinear with respect to the design variables. Numerical values for this problem are shown in Table 4

Two analyses must occur in order to design the two bar truss: a structural analysis and a sizing of the bars constituting the truss. The mass of the bars also provide a load through their weight. Hence, this is a coupled analysis problem since the structural analysis depends on the sizing of each of the bars. The coupled design structure matrix (DSM) is shown in Fig. 5.

The inputs into the design problem are the deterministic and probabilistic parameters of the problem whose values are shown in Table 4. In particular,

$$\mathbf{u_d} = \begin{pmatrix} E & l & r_1 & r_2 & g & h_2 & h_3 \end{pmatrix}^T$$

and

$$\mathbf{u_p} = \begin{pmatrix} \sigma_y & \rho & f \end{pmatrix}^T$$

Table 4. Parameters for the two-bar truss problem.

Parameter	Description	Nominal Value	Distribution
E	Young's Modulus	200×10^6 kN/m^2	–
σ_y	Yield Strength	250×10^3 kN/m^2	$\mathcal{N}(250 \times 10^3, 625 \times 10^6)$
ρ	Density	7850 kg/m^3	$\mathcal{N}(7850, 100)$
l	Length	5 m	–
r_1	Radius of Bar 1	30 mm	–
r_2	Radius of Bar 2	5 mm	–
f	Applied Force	3.5 kN	$\mathcal{N}(3.5, 0.49)$
g	Gravitational Acceleration	9.81 m/s^2	–

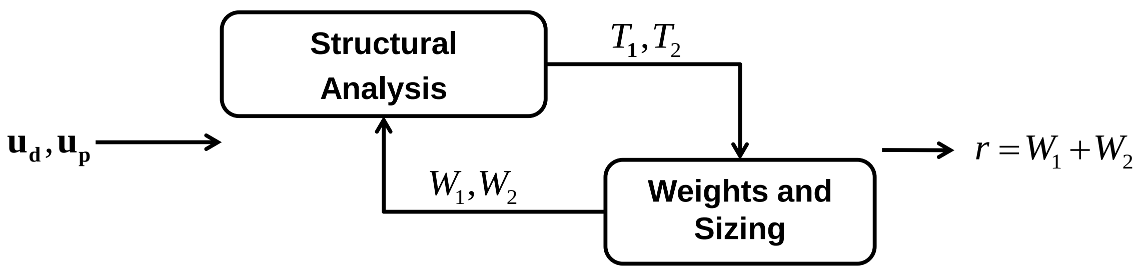


Figure 5. Two bar truss design structure matrix.

The structural analysis CA feeds the forces seen in each of the members of the truss to the weights and sizing module. These can be found through the static equilibrium equations and are found by solving the linear equations

$$\begin{pmatrix} \frac{l}{L_1} & 0 & 1 & 0 & 0 & 0 \\ -\frac{h_2}{L_1} & 0 & 0 & 1 & 0 & 0 \\ -\frac{l}{L_1} & \frac{l}{L_2} & 0 & 0 & 0 & 0 \\ -\frac{h_2}{L_1} & \frac{h_3 - h_2}{L_2} & 0 & 0 & 0 & 0 \\ 0 & \frac{l}{L_2} & 0 & 0 & 1 & 0 \\ 0 & \frac{h_3 - h_2}{L_2} & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ R_{1x} \\ R_{1y} \\ R_{3x} \\ R_{3y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{l}{2h_3}(W_1 + W_2 + 2f) \\ f\left(1 + \frac{h_2}{h_3}\right) \\ 0 \\ 0 \\ f \\ \frac{l}{2h_3}(W_1 + W_2 + 2f) \\ f\left(1 + \frac{h_2}{h_3}\right) + W_1 + W_2 \end{pmatrix}$$

for the tensions. The weights and sizing CA computes the weights of each of the bars based on the nonlinear relationship

$$\mathbf{y_2} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} \pi \rho g r_1^2 L_1 \\ \pi \rho g r_2^2 L_2 \end{pmatrix}$$

Both relationships defined by the CAs rely on the lengths of the bars, which are given by the nonlinear relations

$$L_1 = \sqrt{l^2 + h_2^2}$$
$$L_2 = \sqrt{l^2 + (h_3 - h_2)^2}$$

In order to apply the developed methodology, linearization using a Taylor series expansion about a nominal value (chosen to be the previous iterate's mean value) is conducted. Functionally, this means that the expression for the tensions, $\mathbf{y_1}$, can be expanded as follows

$$\mathbf{y_1} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \approx \begin{pmatrix} \left.\frac{\partial T_1}{\partial \mathbf{u_d}}\right|_{\hat{\mathbf{u}}_\mathbf{d}} (\mathbf{u_d} - \hat{\mathbf{u}}_\mathbf{d}) + \left.\frac{\partial T_1}{\partial \mathbf{u_p}}\right|_{\boldsymbol{\mu}_{\mathbf{u_p}}} (\mathbf{u_p} - \boldsymbol{\mu}_{\mathbf{u_p}}) + \left.\frac{\partial T_1}{\partial \mathbf{y}}\right|_{\hat{\mathbf{y}}} (\mathbf{y} - \hat{\mathbf{y}}) \\ \left.\frac{\partial T_2}{\partial \mathbf{u_d}}\right|_{\hat{\mathbf{u}}_\mathbf{d}} (\mathbf{u_d} - \hat{\mathbf{u}}_\mathbf{d}) + \left.\frac{\partial T_2}{\partial \mathbf{u_p}}\right|_{\boldsymbol{\mu}_{\mathbf{u_p}}} (\mathbf{u_p} - \boldsymbol{\mu}_{\mathbf{u_p}}) + \left.\frac{\partial T_2}{\partial \mathbf{y}}\right|_{\hat{\mathbf{y}}} (\mathbf{y} - \hat{\mathbf{y}}) \end{pmatrix}$$

Similarly, the expression for $\mathbf{y_2}$ can be expanded as

$$\mathbf{y_2} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \approx \begin{pmatrix} \pi \rho g r_1^2 \frac{\hat{y}_2}{\sqrt{l^2 + \hat{y}_2^2}} (h_2 - \hat{y}_2) \\ \pi \rho g r_2^2 \left(\frac{\hat{y}_2 - \hat{y}_3}{\sqrt{l^2 + (\hat{y}_2 - \hat{y}_3)^2}} (h_2 - \hat{y}_2) + \frac{\hat{y}_3 - \hat{y}_2}{\sqrt{l^2 + (\hat{y}_2 - \hat{y}_3)^2}} (h_3 - \hat{y}_3) \right) \end{pmatrix}$$

While this problem is posed as a linear system, the matrix $\mathbf{\Lambda}$ varies with iteration. This requires a Lyapunov analysis to be conducted in order to identify the stability of the system. For this example, this analysis was completed simultaneously with the convergence by numerically solving a matrix Riccati equation. A positive definite matrix, $\mathbf{R}$, for the quadratic problem was able to be found that satisfies the relationship

$$\mathbf{\Lambda}_k^T \mathbf{R}_k \mathbf{\Lambda}_k - \mathbf{R}_k = \mathbf{Q}_k$$

for $\mathbf{Q}_k > 0$. Since a solution for $\mathbf{R}_k$ was able to be found when $\mathbf{Q}_k = \mathbf{I_{4\times 4}}$ for each iterate, this enables a Lyapunov function of the form

$$V_k(\mathbf{y}) = \mathbf{y}_k^T \mathbf{R}_k \mathbf{y}_k$$

to be formed which shows asymptotic stability for a linear, time varying, discrete system. This could also be formulated by using a sum-of-squares Lyapunov function technique as outlined in Ref. 2.

The modulus of the eigenvalues show similar conclusions regarding the asymptotic stability as Lyapunov stability as shown in Fig. 6.

Upon convergence, the value of $\mathbf{y}$, the state variable in the problem, is the mean response for each of the components of the output CAs. In this example, the matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix}$$

since the objective is the weight of truss $W_1 + W_2$, the two elements of the second CA output. Therefore,

$$\bar{r} \approx \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \hat{\mathbf{y}}_{n|n}$$

The estimate for the variance (*i.e.*, the variance bound) in this case is two times the 2-norm of the entire estimated covariance matrix

$$\sigma_r^2 \approx \sum_{i=1}^{2} \| \mathbf{\Sigma}_{\mathbf{y}^*{}_{n|n}} \|_2 = 2 \| \mathbf{\Sigma}_{\mathbf{y}^*{}_{n|n}} \|_2$$

.

Formulating the output of Step 6 in terms of the mean and variance allows for an optimal control problem to be setup for the system's design, where the objective function is defined by

$$\mathcal{J} = \alpha \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \hat{\mathbf{y}}_{n|n} + 2\beta \| \mathbf{\Sigma}_{\boldsymbol{y}^*{}_{n|n}} \|_2$$

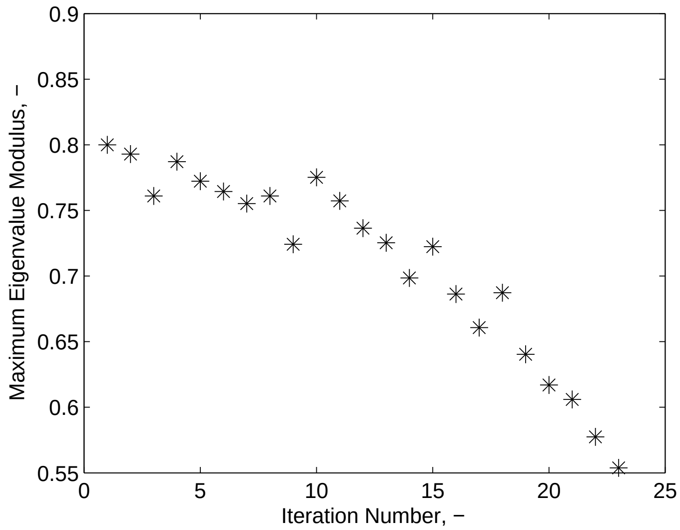


Figure 6. History of the modulus of the maximum eigenvalue of the two bar truss system with iteration.

and α and β allow different weighting on the mean and variance. The nonlinear constraints for this problem as given are

$$\mathbf{g}(\mathbf{y},\mathbf{u}) = \begin{pmatrix} g_1(\mathbf{y},\mathbf{u}) \\ g_2(\mathbf{y},\mathbf{u}) \\ g_3(\mathbf{y},\mathbf{u}) \\ g_4(\mathbf{y},\mathbf{u}) \end{pmatrix} = \begin{pmatrix} |T_1(\mathbf{u})| - \pi r_1^2 \sigma_y \\ |T_2(\mathbf{u})| - \pi r_2^2 \sigma_y \\ -T_1(\mathbf{u}) - \dfrac{\pi^2 E I_1}{L_1^2} \\ -T_2(\mathbf{u}) - \dfrac{\pi^2 E I_2}{L_2^2} \end{pmatrix}$$

Additionally, there is an equality constraint for the control that states

$$\mathbf{h}(\mathbf{y},\mathbf{u}) = \mathbf{u}_k - \mathbf{u}_{k-1} = \mathbf{0}, \quad \forall k \in \{1,\ldots,n\}$$

Approaching this problem as a discrete optimal control problem, the Hamiltonian is given by

$$\begin{aligned} H(\mathbf{y},\mathbf{u}) = \psi_0 \left(\alpha \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \hat{\mathbf{y}}_{n|n} + 2\beta \parallel \mathbf{\Sigma}_{\boldsymbol{y^*}_{n|n}} \parallel_2 \right) \\ + \boldsymbol{\gamma}^T \left(\mathbf{u}_k - \mathbf{u}_{k-1} \right) + \boldsymbol{\lambda}^T \begin{pmatrix} |T_1(\mathbf{u})| - \pi r_1^2 \sigma_y \\ |T_2(\mathbf{u})| - \pi r_2^2 \sigma_y \\ -T_1(\mathbf{u}) - \dfrac{\pi^2 E I_1}{L_1^2} \\ -T_2(\mathbf{u}) - \dfrac{\pi^2 E I_2}{L_2^2} \end{pmatrix} \end{aligned}$$

where ψ_0, $\boldsymbol{\gamma}$, and $\boldsymbol{\lambda}$ are Lagrange multipliers. When the terms in this relation are computed numerically, the optimal control conditions can then be used in SNOPT (or another NLP solver) to compute the values of h_2 and h_3 for a chosen value of α and β.

Note, that in this problem, continuation was used to obtain the solution. Each of the constraints were applied in order with the prior solution seeding the solution to the next solution. The optimization order was as follows:

1. Unconstrained optimization
2. Bar 1 yield constraint
3. Bar 1 yield and buckling constraints

4. Bar 1 yield and buckling constraint and bar 2 yield constraint
5. Bar 1 yield and buckling constraints and bar 2 yield and buckling constraints.

V.B.1. Design Results

The formulation of this problem is based on work by Ref. 10 that obtained the deterministic optimal design as shown in Fig. V.B.1. This figure also shows the minimum mass and minimum variance robust designs found in the present investigation. The Ref. 10 solution has (h_2^*, h_3^*) at $(0.751, 9.970)$ m with an objective function value of $\mathcal{J} = 291.0921$ N. The deterministic case obtained in the present investigation yields a very similar result with (h_2^*, h_3^*) at $(0.746, 9.991)$ m with an objective function value of $\mathcal{J} = 291.151$ N demonstrating that the method developed achieves an accurate numerical result even in the case of nonlinear problems.

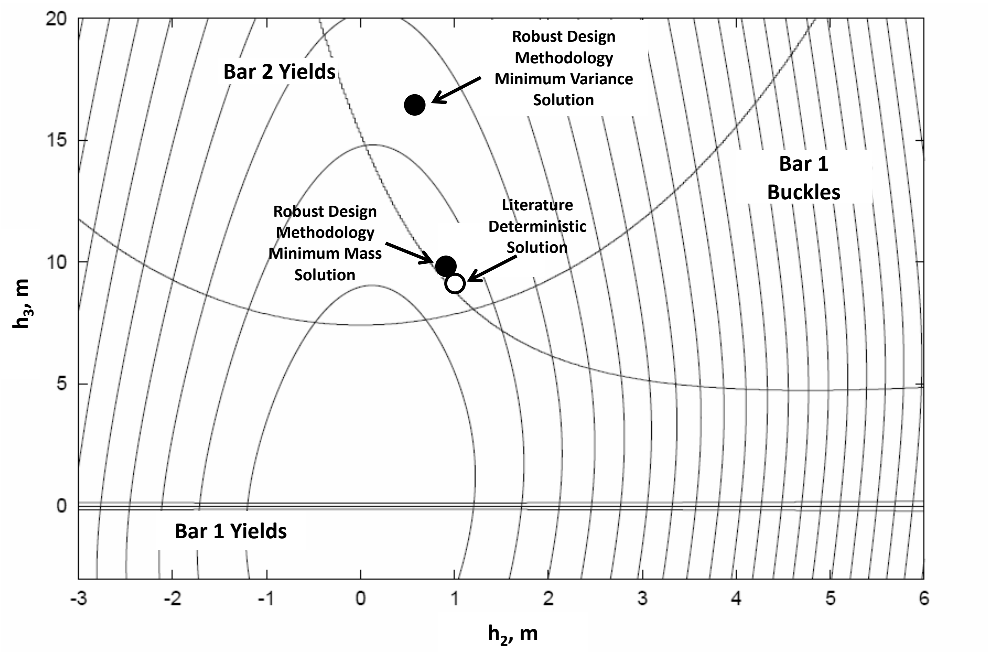


Figure 7. Design solutions for the two-bar truss showing the design space with the optimal designs.

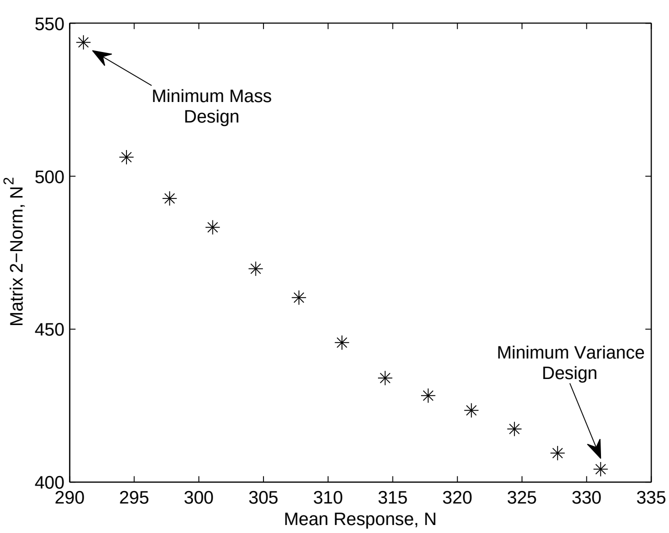


Figure 8. Design solutions for the two-bar truss showing the variation of $2||\mathbf{\Sigma_{y^*}}||_2$ with the mean objective function.

A Pareto-front in terms of the mean weight and variance in the weight for this problem is shown in Fig. V.B.1. From this figure, the deterministic optimum provides the minimum mean response; however, it is not the minimum variance solution. This minimum variance design is approximately 39 N heavier.

VI. Conclusions

The use of control theory with the multidisciplinary design problem was demonstrated through the enforcement of inequality and equality constraints on the design. This technique was shown to be able to accommodate constraints that are functions of the design variables and CA output. These are considered at the same level of the optimization/convergence hierarchy, which allows for an efficient coordinated search of the design space. Established solution techniques for the discrete-time optimal control problem through NLP methods enable design solutions to be found efficiently. In addition, in some instances continuous optimal control techniques such as indirect methods are applicable to the design problem. Finally, approaching the design constraint problem as an optimal control problem enables continuation techniques to be applied to design variables which may enable design solutions to be found in difficult design spaces.

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