Design Convergence Using Stability Concepts from Dynamical Systems Theory

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The inherent iteration required in the multidisciplinary design problem allows the design problem to cast as a dynamical system. The iteration in design is a resultant of the two root-finding problems. The first root-finding problem is in seeking out candidate designs while the second is in optimizing the candidate designs. Viewing the root-finding schema as a dynamical system allows the application of established techniques from dynamical systems theory to design. Stability theory is one of the techniques that is enabled by viewing multidisciplinary design as a dynamical system. Stability theory is capable of providing information on whether or not a design will converge for a given iteration scheme, starting values for the iteration that will lead to convergence, as well as information regarding how fast a design will converge. Following the theoretical development, each of these concepts is demonstrated on sample problems showing the benefit of the application of stability theory in the design realm.

Nomenclature

$(\cdot)_e$	Equilibrium of $(\cdot)$
$(\cdot)^*$	Root value of $(\cdot)$
$(\cdot)_k$	Iterate k of $(\cdot)$
$\boldsymbol{\lambda}$	Lagrange multiplier
$\boldsymbol{\Phi}(k,j)$	Discrete state transition matrix from iterate k to iterate j
$\mathbb{C}$	Set of complex numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{Z}$	Set of all integers
$\mathbb{Z}_+$	Set of all positive integers (*i.e.*, $\mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}$)
$\mathbf{f}(\mathbf{u},\mathbf{p})$	Contributing analysis function mapping
$\mathbf{g}_i(\mathbf{u},\mathbf{p})$	Inequality constraints
$\mathbf{h}_i(\mathbf{u},\mathbf{p})$	Equality constraints
$\mathbf{p}$	Design parameter
$\mathbf{u}$	Design variables
$\mathbf{x}$	The state, the output of the contributing analysis
$\mathbf{z}$	Column vector of monomials used in sum-of-squares
$\mathbf{z}$	Requirement value
$\mathcal{A}$	Region of attraction
$\mathcal{J}(\mathbf{u},\mathbf{p})$	Objective function
$\{\lambda_i\}$	Set of eigenvalues
$L(\cdot)$	Lagrangian
MDA/O	Multidisciplinary analysis/optimization

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I. Introduction

Complex system design is comprised of analyses from numerous disciplines. When each of the disciplines use the same information, have a common set of assumptions, and satisfy the constraints imposed on the design, the design is said to be converged. The convergence process for complex, multidisciplinary designs is typically lengthy without and is typically undertaken without the knowledge whether a converged design even exists. Dynamical systems theory is a well researched field and there is an inherent analogue between the multidisciplinary design problem and dynamical systems theory.[1–7] In Ref. 8, formalism and example applications of casting the multidisciplinary design problem as a dynamical system is provided and it is shown that design is fundamentally a root-finding problem, which can be thought of as a discrete dynamical system.

This work discusses some of the theoretical constructs required to view the design problem as a dynamical system. It then builds upon these foundations to provide a specific application of how dynamical system theory can be used in the convergence process of multidisciplinary design. In particular, concepts from the stability domain of dynamical system theory are applied to multidisciplinary design in order to identify:

1. Whether a feasible design exists (for a given iteration scheme)
2. Whether an optimal design exists (for a given iteration scheme)
3. The range of initial values that can be used to converge the design
4. The rate at which the design will converge

Each of these is an enhancement compared to current multidisciplinary design and analysis (MDA/O) techniques enabled by viewing iterative relationships formed in the convergence of the design problem as a dynamical system.

II. Viewing the Multidisciplinary Design Problem as a Dynamical System

II.A. Identification of Candidate Designs

Identifying candidate designs in multidisciplinary systems can be thought of as the process of finding the root of a function. Consider a multidisciplinary problem where the analysis variables are described by a multivariable function $\mathbf{f}(\mathbf{u}, \mathbf{p})$ where $\mathbf{u}$ are the design variables and $\mathbf{p}$ are parameters of the problem. Assume that the requirements of the design are given by only equality constraints that are a function of the performance of the system. The performance of the design is described by a multi-variable mapping $\mathbf{g}(\mathbf{f}(\mathbf{u}, \mathbf{p}))$ and the requirements are given by $\mathbf{z}$. In order to meet the requirements it is necessary to adjust the design variables $\mathbf{u}$ so that

$$\mathbf{z} = \mathbf{g}(\mathbf{f}(\mathbf{u}, \mathbf{p})) \tag{1}$$

Equation (1) can be rewritten as

$$\mathbf{z} - \mathbf{g}(\mathbf{f}(\mathbf{u}, \mathbf{p})) = \mathbf{0} \tag{2}$$

The solution $\mathbf{u}^*$ of Eq. (2) is the root of the system and the process is referred to as root-finding. Since identifying feasible designs within the multidisciplinary design problem requires finding the value of $\mathbf{u}$ that satisfies Eq. (2), this process can be thought of as a root-finding problem when an iterative solution method is chosen.

Many numerical methods for finding the root of a function, $\mathbf{g}(\mathbf{x})$, are dynamical systems since they rely on iterative schemes to identify the root.[1] For instance, the bisection method, secant method, function iteration method, and Newton's method are all iterative techniques that satisfy the requirements of a dynamical system. The choice of iteration method may affect the convergence characteristics of the system.

II.B. Design Optimization

In order for a candidate design to be an optimum with respect to some objective function, its performance needs to be evaluated with respect to other potential designs.

The first-order, necessary condition associated with optimization problem given by

$$\left.\begin{array}{ll}\textbf{Minimize:} & \mathcal{J}(\mathbf{u},\mathbf{p}) \\ \textbf{Subject to:} & \mathbf{g}_i(\mathbf{u},\mathbf{p}) \leq \mathbf{0}, \quad i=1,\ldots,n_g \\ & \mathbf{h}_j(\mathbf{u},\mathbf{p}) = \mathbf{0}, \quad j=1,\ldots,n_h \\ \textbf{By varying:} & \mathbf{u}\end{array}\right\} \tag{3}$$

require the a stationary point of the Lagrangian to be defined. For the optimization problem given in Eq. (3), the Lagrangian is

$$L(\mathbf{u},\mathbf{p},\boldsymbol{\lambda}) = \mathcal{J}(\mathbf{u},\mathbf{p}) + \sum_{i=1}^{n_g} \lambda_i \mathbf{g}_i(\mathbf{u},\mathbf{p}) + \sum_{j=1}^{n_h} \lambda_{n_g+j} \mathbf{h}_j(\mathbf{u},\mathbf{p}) \tag{4}$$

The first-order, necessary conditions for $\mathbf{u}^*$ to be an optimum are[9]

1. $\mathbf{u}^*$ is feasible
2. $\lambda_i \mathbf{g}_i(\mathbf{u}^*,\mathbf{p}) = \mathbf{0} \quad i=1,\ldots,n_g$ and $\lambda_i \geq 0$
3. $\nabla_{\mathbf{u}} L(\mathbf{u},\mathbf{p},\lambda) = \nabla_{\mathbf{u}} \mathcal{J}(\mathbf{u},\mathbf{p}) + \sum_{i=1}^{n_g} \lambda_i \nabla_{\mathbf{u}} \mathbf{g}_i(\mathbf{u},\mathbf{p}) + \sum_{j=1}^{n_h} \lambda_{n_g+j} \nabla_{\mathbf{u}} \mathbf{h}_j(\mathbf{u},\mathbf{p}) = \mathbf{0}$ with all $\lambda_i \geq 0$ and λ_{n_g+j} unrestricted in sign

Each of the necessary conditions can be considered a root-finding problem by itself. Thus, the optimization process can also be considered as a root-finding problem.

II.C. Identifying an Optimal Multidisciplinary Design

Multidisciplinary design optimization can be broken down into two steps: (1) identifying feasible designs and (2) identifying the optimal design from the set of feasible candidates. As discussed, both of these steps are root-finding problems. With the choice of an appropriate iterative numerical root-finding scheme, each of these individual steps can be posed as dynamical systems. When combined together, a nested root-finding problem results, whereby the function being optimized is actually a root-finding problem itself. This is shown in Fig. 1

Since both steps of this process are numerical root-finding problems that are described by an autonomous, discrete dynamical system, the theoretical development that follows considers a general autonomous, discrete dynamical system and the example problems demonstrate the use of stability in the convergence process.

III. Stability Analysis

The concept of stability allows for the identification of feasible designs for given iteration schemes. These iteration schemes can usually be written in the form

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k) \tag{5}$$

where $\mathbf{x}$ is the state of the system, $\mathbf{f}$ is a function which describes the time evolution of the system, $\mathbf{u}$ is the input into the system, and k is the iterate number. A specific instance of Eq. (5) is a linear, discrete dynamical system, which is given by

$$\mathbf{x}_{k+1} = \mathbf{A}_k \mathbf{x}_k + \mathbf{B}_k \mathbf{u}_k \tag{6}$$

For a given initial state, a system is stable if the state does not grow beyond the initial state's magnitude. More rigorously, this is defined in terms of equilibrium points of a system. Consider the discrete dynamical system defined by Eq. (5), the equilibrium point is defined as

For a linear dynamical system, given by Eq. (6), the equilibrium point is only the origin of the system (*i.e.*, $\mathbf{x}_e = \mathbf{0}$).

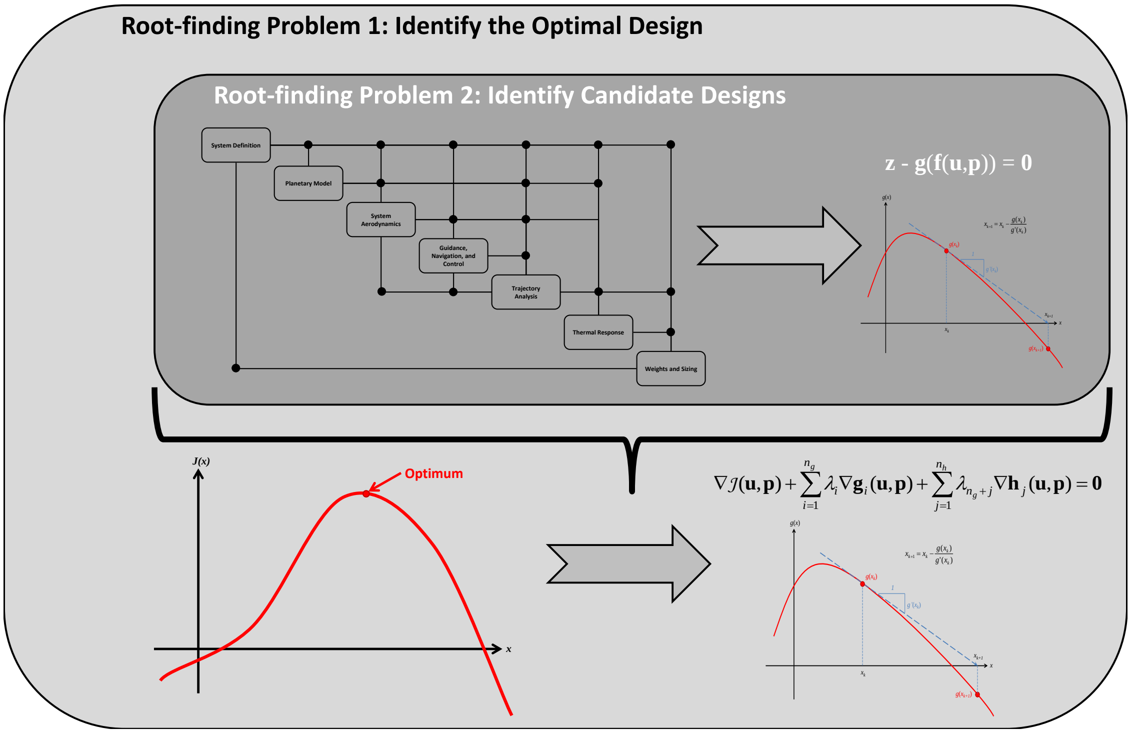


Figure 1. Multidisciplinary design through root-finding.

Definition: Equilibrium of a Dynamical System

A particular point $\mathbf{x}_e$ is an *equilibrium point* of the dynamical system given by Eq. (5) if the system's state at iterate $k = 0$ is $\mathbf{x}_e$ and $\forall k \in \mathbb{Z}_+ \setminus \{0\}$, $\mathbf{f}(\mathbf{x}_e, \mathbf{0}) = \mathbf{x}_e$.

The equilibrium point's stability is defined with regard to the zero-input discrete dynamical system given by[2–5]

$$\left.\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{f}(\mathbf{x}_k, \mathbf{0}) \\ \mathbf{x}_{k=0} &= \mathbf{x}_0 \end{aligned}\right\} \tag{7}$$

Definition: Stability

For the system given by Eq. (7), if $\forall \epsilon > 0$, $\exists \delta(\epsilon, 0) \in (0, \epsilon]$ an equilibrium point of the system is

- *stable* if $\forall k > 0$ and $\| \mathbf{x}_0 \| < \delta$, $\| \mathbf{x}_k \| < \epsilon$
- *asymptotically stable* if
  1. the equilibrium point is stable and
  2. $\exists \delta' \in (0, \epsilon]$ such that whenever $\| \mathbf{x}_0 \| < \delta'$ the state's evolution satisfies $\lim_{k \to \infty} \| \mathbf{x}_k \| = 0$
- *unstable* if it is not stable or asymptotically stable

Figures 2 and 3 demonstrate the concept of equilibrium point stability. Figure 2 shows a more intuitive concept of stability while Fig. 3 demonstrates different state trajectories in $\mathbb{R}^2 \times \mathbb{R}$ and $\mathbb{R}^2$.

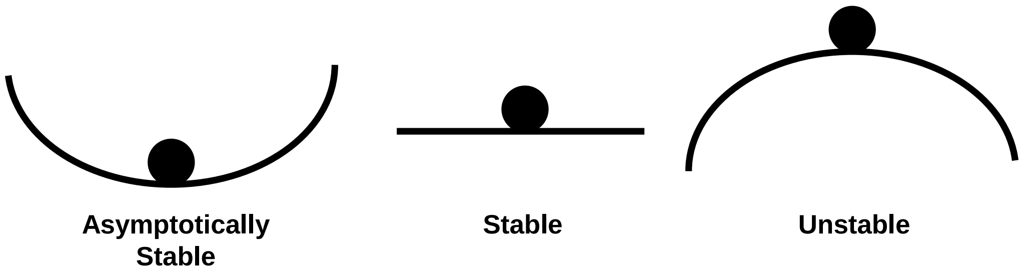


Figure 2. Visualization of the concept of stability.

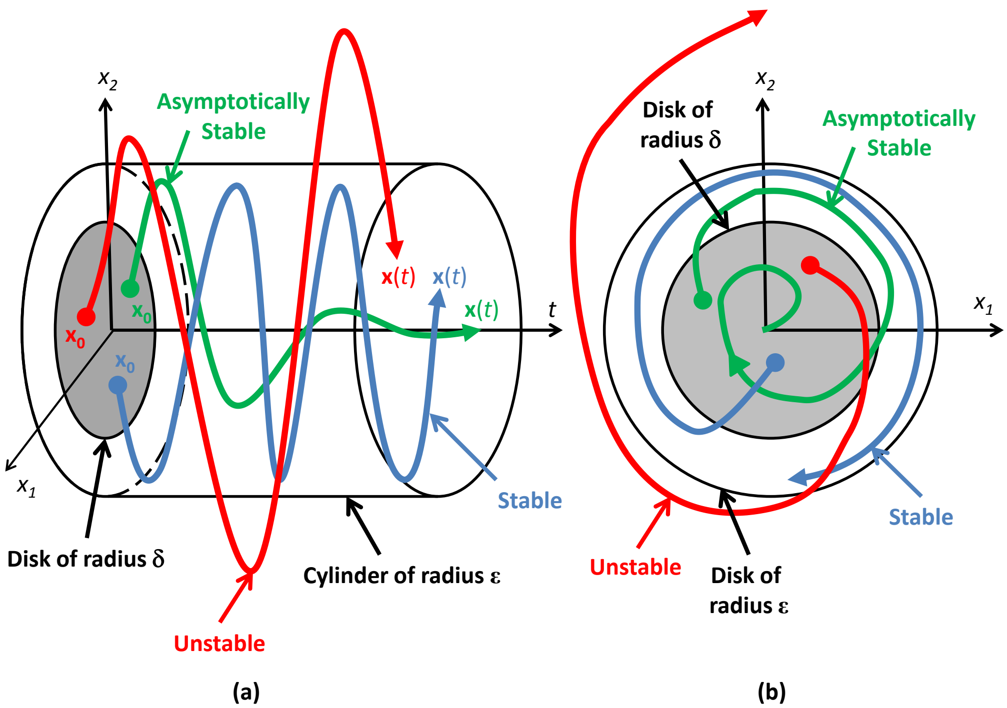


Figure 3. Visualization of state trajectories in (a) $\mathbb{R}^2 \times \mathbb{R}$ and (b) $\mathbb{R}^2$ showing stability for a continuous dynamical system.

III.A. Linear Stability Criterion

For discrete, linear systems, that is dynamical systems given by Eq. (6), the solution for the evolution of the state and the output is given by

$$\mathbf{x}_k = \mathbf{\Phi}(k,0)\mathbf{x}_0 + \sum_{j=1}^{k} \mathbf{\Phi}(k,j)\mathbf{B}_{j-1}\mathbf{u}_{j-1} \tag{8}$$

where $\mathbf{\Phi}(k,j)$ is the discrete state transition matrix. This transition matrix is given by

$$\mathbf{\Phi}(k,j) = \mathbf{A}^{k-j} \tag{9}$$

in the case where $\mathbf{A}_k = \mathbf{A}\ \forall k \in \mathbb{Z}_+$, that is when $\mathbf{A}$ is constant. Substituting Eq. (8) and Eq. (9) into Eq. (6) yields[2]

$$\mathbf{x}_{k+1} = \mathbf{A}^{k+1}\mathbf{x}_0 + \sum_{j=1}^{k} \mathbf{A}^{k-j+1}\mathbf{B}_{j-1}\mathbf{u}_{j-1} + \mathbf{B}_k\mathbf{u}_k \tag{10}$$

which is a relationship that depends on the initial condition and the control history. In the unforced case (*i.e.*, $\mathbf{u}_k = \mathbf{0}\ \forall k \in \mathbb{Z}_+$) and by the Cayley-Hamilton theorem, the stability criterion can be identified. If $\max_i |\lambda_i| > 1$ for any simple root of the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{11}$$

or $\max_i |\lambda_i| \geq 1$ for any repeated root of Eq. (11) then the system is unstable. This is because the Jordan canonical form of $\mathbf{A}$ has terms terms that tend to infinity as the iteration proceeds (*i.e.*, $\lim_{k\to\infty} \mathbf{V}^T\mathbf{A}^k\mathbf{V} = \infty$ since diagonal terms of $\mathbf{V}^T\mathbf{A}\mathbf{V}$ are greater than unity). Similarly, if $\max_i |\lambda_i| \leq 1$ for any simple root or $\max_i |\lambda_i| < 1$ for repeated roots of Eq. (11), then the iteration scheme is asymptotically stable.[2,6,7] More rigorous proof of this concept is provided in Ref. 6.

III.B. Lyapunov Stability

Stability of general dynamical systems, including the one formed for design, can be studied using Lyapunov stability theory. This theory lays the foundations to assess the stability characteristics of arbitrary designs and can be leveraged to give additional characteristics about the convergence properties of the design. For instance, it can be used to ascertain information regarding the convergence rate and what starting iteration values will lead to a converged design, for a given root-finding scheme.

Lyapunov stability theory is prevalent for continuous dynamical systems such as the autonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \forall t \in [0, \infty) \tag{12}$$

for which the origin is an equilibrium point, a Lyapunov function is a continuously differentiable map $V : \mathbb{R}^n \to \mathbb{R}$ such that

1. $V(\mathbf{x}) > 0,\ \mathbf{x} \neq \mathbf{0},\ V(\mathbf{0}) = 0$
2. $\frac{d}{dt}\left(V(\mathbf{x}(t))\right) \leq 0, \quad \forall t \in [0, \infty)$

where $\mathbf{x} : [0, \infty) \to \mathbb{R}^n$ is any solution of Eq. (12).[4] In fact, it has been applied to differential equations such as this since Lyapunov in 1892. However, its use in dynamical systems defined by difference equations, such as those used to converge and optimize designs, is less mature with the first treatment in the literature being attributed to Hahn in 1958.[10]

To begin the development of Lyapunov theory for discrete dynamical systems (such as those defined in design), consider the following definition of a Lyapunov function[5,7,10–12]

Definition: Discrete Lyapunov Function

A mapping $V : \mathbb{R}^n \to \mathbb{R}$ is a *Lyapunov function* for the zero-input autonomous, discrete dynamical system, Eq. (5), (*i.e.*, $\mathbf{f}(\mathbf{x}_k, \mathbf{0})$) at an equilibrium point $\mathbf{x}_e$ of $\mathbf{f}$ if there is an open neighborhood $\mathcal{D}$ at $\mathbf{x}_e$ such that V is continuous on $\mathcal{D}$ and

- $V(\mathbf{x}) > 0\ \forall \mathbf{x} \in \mathcal{D},\ \mathbf{x} \neq \mathbf{x}_e,\ V(\mathbf{x}_e) = 0$
- $\Delta V = V(\mathbf{x}_{k+1}) - V(\mathbf{x}_k) \leq 0$ whenever $\mathbf{x}_k, \mathbf{x}_{k+1} \in \mathcal{D}$

With this definition, the following theorem can be presented.

Theorem 1 (Lyapunov's Direct Method for Discrete Dynamical Systems)**.** *Consider the following dynamical system*

$$\left.\begin{array}{l}\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k), \quad \mathbf{x}_k \in \mathcal{S} \subseteq \mathcal{D} \\ \mathbf{f}(\mathbf{0}) = \mathbf{0}\end{array}\right\}$$

where it is assumed that $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ *is continuous on an open neighborhood* $\mathcal{S}$ *of a fixed point* $\mathbf{x}_e$ *and that* $V : \mathbb{R}^n \to \mathbb{R}$ *is a Lyapunov function for* $\mathbf{f}$ *at* $\mathbf{u}^*$*, then at* $\mathbf{u}^*$ *the dynamics governed by* $\mathbf{f}$ *is* stable. *If, in addition,*

$$\Delta V = V(\mathbf{x}_{k+1}) - V(\mathbf{x}_k) < 0 \textit{ whenever } \mathbf{x}, \mathbf{x}_{k+1} \in \mathcal{D} \textit{ and } \mathbf{x}_k \neq \mathbf{x}_e$$

then the trajectory governed by $\mathbf{f}$ *is* asymptotically stable *at* $\mathbf{x}_e$*. If* $\mathcal{S} = \mathcal{D} = \mathbb{R}^n$ *and*

$$V(\mathbf{x}_k) \to \infty \textit{ as } \| \mathbf{x}_k \| \to \infty,$$

then the dynamics governed by $\mathbf{f}$ *is* globally asymptotically stable *at* $\mathbf{x}_e$*.*

The proof of this theorem is shown in the appendix.

A special case of a discrete dynamical system is that of a linear, discrete system with constant coefficients such as that shown in Eq. (6) with $\mathbf{A}_k = \mathbf{A}\ \forall k \in \mathbb{Z}_+$. The zero-input stability in this case can be investigated using a quadratic Lyapunov function of the form

$$V(\mathbf{x}) = \mathbf{x}^T \mathbf{R} \mathbf{x} \tag{13}$$

This form leads to

$$\Delta V(\mathbf{x}) = V(\mathbf{x}_{k+1}) - V(\mathbf{x}_k) = \mathbf{x}^T \left(\mathbf{A}^T \mathbf{R} \mathbf{A} - \mathbf{R} \right) \mathbf{x} = -\mathbf{x}^T \mathbf{S} \mathbf{x} \tag{14}$$

For any given $\mathbf{S} > 0$, which is symmetric there is exactly one solution for a symmetric matrix $\mathbf{R}$ which is the solution of Stein's equation

$$\mathbf{A}^T \mathbf{R} \mathbf{A} - \mathbf{R} = -\mathbf{S} \tag{15}$$

provided that

$$\lambda_i \neq \lambda_j \neq 1, \quad i = 1, 2, \cdots, n, \ j = 1, 2, \cdots, n \tag{16}$$

holds for all eigenvalues λ_i of $\mathbf{A}$. Thus, if there is a solution, $\mathbf{R}$, to Stein's equation, Eq. (15), then the linear system is globally asymptotically stable since $\Delta V < 0$, $\mathcal{S} = \mathcal{D} = \mathbb{R}^n$, and $V(\mathbf{x}_k) \to \infty$ as $\| \mathbf{x}_k \| \to \infty$. Note that this is equivalent to the results before (*i.e.*, if an $\mathbf{R}$ exists, this implies $|\lambda_i| < 1$ for all eigenvalues).

III.C. Summary of Stability Conditions

A summary of the conditions to achieve stability for both a general dynamical system (in terms of Lyapunov functions) and linear, constant coefficient systems (in terms of eigenvalue criterion) are listed in Table 1.

Table 1. Discrete dynamical system stability criterion.[2,6,7]

Classification	**General System Criterion**	**Linear Constant System Criterion**
Unstable		If $\lvert\lambda_i\rvert > 1$ for any simple root or $\lvert\lambda_i\rvert \geq 1$ for any repeated root
Stable	1. $V(\mathbf{x}) > 0$ 2. $\Delta V \leq 0$	If $\lvert\lambda_i\rvert \leq 1$ for any simple root and $\lvert\lambda_i\rvert < 1$ for all repeated roots
Asymptotically Stable	1. $V(\mathbf{x}) > 0\ \forall \mathbf{x} \neq \mathbf{0}$ and $V(\mathbf{0}) = 0$ 2. $\Delta V < 0\ \forall \mathbf{x} \neq \mathbf{0}$ (or $\Delta V \leq 0\ \forall \mathbf{x}$ and $\Delta V \neq 0$ for any solution sequence $\{\mathbf{x}_k\}$)	$\lvert\lambda_i\rvert < 1$ for all roots (or $\exists \mathbf{R}$ that satisfies $\mathbf{A}^T\mathbf{R}\mathbf{A} - \mathbf{R} = -\mathbf{S}$ with $\mathbf{S} = \mathbf{S}^T > 0$)
Globally Asymptotically Stable	1. $V(\mathbf{x}) > 0\ \forall \mathbf{x} \neq \mathbf{0}$ and $V(\mathbf{0}) = 0$ 2. $\Delta V < 0\ \forall \mathbf{x} \neq \mathbf{0}$ (or $\Delta V \leq 0\ \forall \mathbf{x}$ and $\Delta V \neq 0$ for any solution sequence $\{\mathbf{x}_k\}$) 3. $V(\mathbf{x}) \to \infty$ as $\| \mathbf{x} \| \to \infty$	

IV. Stability and Its Relation to Design Convergence

From the multidisciplinary design perspective, stability of the dynamical system gives information into the convergence characteristics of the design. Asymptotic stability implies that there is a limited region for which the design will converge whereas global asymptotic stability implies that the design will converge with enough iteration regardless of the design assumptions used to start the convergence procedure. If the dynamical system representing the multidisciplinary design is unstable or stable it implies that the design will not converge. This is an analogous to the requirement that a contraction mapping exist.

V. Region of Attraction

The region of attraction to an equilibrium point $\mathbf{x}_e$ of Eq. (7) is the set

$$\mathcal{A} = \{\mathbf{x} : \mathbf{f}^k(\mathbf{x}) \to \mathbf{x}_e \text{ as } k \to \infty\}$$

This can be more readily understood as the set of initial guesses that make the iteration scheme converge to a design. The following theorem helps in identifying this region of attraction[13]

Theorem 2 (Region of Attraction)**.** *Assume that* $\phi : \mathbb{R}^n \to \mathbb{R}$ *is continuous and satisfies*

1. $\phi(\mathbf{x}_e) = 0$
2. $\phi(\mathbf{x}),\ \mathbf{x} \neq \mathbf{x}_e$
3. $\phi(\mathbf{x}) \geq a$ *for* $\| \mathbf{x} - \mathbf{x}_e \| \geq b$

where a *and* b *are positive constants and* $\mathbf{x}_e$ *is a fixed point of* $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$*. Assume also* $\exists w : \mathbb{R}^n \to \mathbb{R}$ *is continuous at* $\mathbf{x}_e$ *with*

1. $w(\mathbf{x}_e) = 0$
2. $w(\mathbf{x}) > 0,\ \mathbf{x} \neq \mathbf{x}_e$
3. $w(\mathbf{f}(\mathbf{x})) - w(\mathbf{x}) = -\phi(\mathbf{x})(1 - w(\mathbf{x}))\ \forall \mathbf{x} \in \mathbb{R}^n$

Then $\mathcal{A}_0 = \{\mathbf{x} : w(\mathbf{x}) < 1\}$ *is the region of attraction,* $\mathcal{A}$*.*

A function of the form $\phi(\mathbf{x}) = c \| \mathbf{x} - \mathbf{x}_e \|^p$ satisfies the three required conditions for ϕ. Therefore, the problem of finding the domain of attraction becomes a problem of finding the domain for w such that $w(\mathbf{x}) < 1$.

VI. Seeking Lyapunov Functions

In general the search of a Lyapunov function $V(\mathbf{x})$ is a difficult one, particularly for nonlinear systems for which the equations describing their evolution may not be known, as would likely be the case in design. However, several techniques exist for their search exist.[3–7] An emerging technique that is used in this work to identify Lyapunov functions is sum-of-squares decomposition. This technique is particularly applicable for polynomial dynamical systems (including Taylor series approximations) and achieves a Lyapunov function by factoring an nonlinear polynomial that is parameterized by unknown variables into a sum-of-squares. The resulting sum-of-squares polynomial positive definite and can be used to check the difference condition to find if iteration schema for the design is convergent.

VI.A. Sum-of-squares Decomposition and Analysis

A multivariate polynomial, $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$ is said to be a sum-of-squares if there exist polynomials $f_1(\mathbf{x})$,..., $f_m(\mathbf{x})$ such that

$$f(\mathbf{x}) = \sum_{i=1}^{m} f_i^2(\mathbf{x}) \tag{17}$$

This statement is equivalent to the following proposition.[14]

Proposition 1. *Let $f(\mathbf{x})$ be a polynomial in $\mathbf{x} \in \mathbb{R}^n$ of degree $2d$. In addition, let $\mathbf{z}(\mathbf{x})$ be a column vector whose entries are all monomials in $\mathbf{x}$ with degree no greater than d. Then $f(\mathbf{x})$ is a sum-of-squares if and only if there exists a positive semi-definite matrix $\mathbf{Q}$ such that*

$$f(\mathbf{x}) = \mathbf{z}^T(\mathbf{x})\mathbf{Q}\mathbf{z}(\mathbf{x}) \tag{18}$$

With this definition, it can be seen that a sum-of-squares decomposition can be found using semidefinite programming, to search for the $\mathbf{Q}$ matrix satisfying Eq. (18).

What is significant about sum-of-squares decomposition for design applications, is that it allows the search of a polynomial Lyapunov function $V(\mathbf{x})$ (*i.e.*, the $f(\mathbf{x})$) to have coefficients that are parameterized in terms of some other unknowns. A search for the coefficients that render the polynomial $f(\mathbf{x})$ a sum-of-squares can still be performed using semidefinite programming. For example, consider the construction of a Lyapunov function for a nonlinear system where the following procedure can be used:

1. Coefficients can be used to parameterize a set of candidate Lyapunov functions in an affine manner, that is it can determine a set $\mathcal{V} = \{V(\mathbf{x}) : V(\mathbf{x}) = v_0(\mathbf{x}) + \sum_{i=1}^{m} c_i v_i(\mathbf{x})\}$, where the $v_i(\mathbf{x})$'s are monomials in $\mathbf{x}$.
2. Search for a function $V(x) \in \mathcal{V}$ which satisfies $V(\mathbf{x}) - \phi(\mathbf{x})$ and $-\dfrac{\partial V(\mathbf{x})}{\partial \mathbf{x}} f(\mathbf{x})$, where $\phi(\mathbf{x}) > 0$ using semidefinite programing

The semidefinite programming problem above determines the state dependent linear matrix inequalities (LMIs) that govern the problem which are resultants of solving the the following convex optimization problem

$$\left.\begin{array}{ll} \textbf{Minimize}: & \displaystyle\sum_{i=1}^{m} a_i c_i \\ \textbf{Subject to}: & \mathbf{F}_0(\mathbf{x}) + \sum_{i=1}^{m} c_i \mathbf{F}_i(\mathbf{x}) \geq 0 \\ \textbf{By varying}: & c_i \end{array}\right\} \tag{19}$$

where $a_i \in \mathbb{R}$ are fixed coefficients, $c_i \in \mathbb{R}$ are decision variables, and $\mathbf{F}_i(\mathbf{x})$ are symmetric matrix functions of the indeterminate $\mathbf{x} \in \mathbb{R}^n$. When $\mathbf{F}_i(\mathbf{x})$ are symmetric polynomial matrices in $\mathbf{x}$ the computationally difficult problem of solving (19) is relaxed according to the following proposition[14]

Proposition 2. *Let $\mathbf{F}(\mathbf{x})$ be an $m \times m$ symmetric polynomial matrix of degree $2d$ in $\mathbf{x} \in \mathbb{R}^n$. Furthermore, let $\mathbf{Z}(\mathbf{x})$ be a column vector whose entries are all monomials in $\mathbf{x}$ with degree no greater than d, and assume the following:*

(i) $\mathbf{F}(\mathbf{x}) \geq 0 \quad \forall \mathbf{x} \in \mathbb{R}^n$

(ii) $\mathbf{v}^T\mathbf{F}(\mathbf{x})\mathbf{v}$ *is a sum of squares, with* $\mathbf{v} \in \mathbb{R}^m$

(iii) There exists a positive semi-definite matrix $\mathbf{Q}$ *such that* $\mathbf{v}^T\mathbf{F}(\mathbf{x})\mathbf{v} = (\mathbf{v} \otimes \mathbf{Z}(\mathbf{x}))^T \mathbf{Q} (\mathbf{v} \otimes \mathbf{Z}(\mathbf{x}))$

Then (i) $\Leftarrow$ (ii) and (ii) $\Leftrightarrow$ (iii)

This proposition is proven by Prajna *et al.* in Ref. 14. However, by applying Proposition 1, it is seen that the solution to the sum-of-squares optimization problem seen in Eq. (20) is also a solution to the state-dependent LMI problem, Eq. (19).

$$\left.\begin{array}{ll} \textbf{Minimize}: & \displaystyle\sum_{i=1}^{m} a_i c_i \\ \textbf{Subject to}: & \mathbf{v}^T \left(\mathbf{F}_0(\mathbf{x}) + \displaystyle\sum_{i=1}^{m} c_i \mathbf{F}_i(\mathbf{x})\right) \mathbf{v} \text{ is a sum-of-squares polynomial} \\ \textbf{By varying}: & c_i \end{array}\right\} \tag{20}$$

This relaxation of the LMI problem turns the relatively difficult computation problem associated with Eq. (19) to a relatively simple computational problem since semidefinite programming solvers are readily available on multiple platforms.[15,16]

VII. Estimating the Rate of Convergence Based on Lyapunov-like Techniques

For a special case of asymptotically stable systems, the rate of convergence can be estimated—that of exponentially stable systems. The following lemma defines the basis of exponential stability for a discrete dynamical system

Lemma 1. *For a system defined by Eq. (5) if there exists a function* $V(\mathbf{x})$ *with* $V(\mathbf{0}) = 0$ *such that*

1. $V(\mathbf{x}_k) \geq c\phi(\parallel \mathbf{x}_k \parallel)$
2. $\Delta V = V(\mathbf{x}_{k+1}) - V(\mathbf{x}_{k+1}) \leq M - \alpha V(\mathbf{x}_k)$

for some $\phi \in \mathcal{K}$ *and constants* $c > 0$*,* $M \geq 0$*, and* $0 < \alpha < 1$ *then*

1. $c\phi(\parallel \mathbf{x}_k \parallel) \leq V(\mathbf{x}_k) \leq (1-\alpha)^k V(\mathbf{x}_0) + M \sum_{i=0}^{k-1} (1-\alpha)^i$
2. $\lim_{k \to \infty} \phi(\parallel \mathbf{x}_k \parallel) \leq \dfrac{M}{c\alpha}$

The proof of this lemma is found by application of a geometric series as shown in Ref. 17. The two conclusions of this lemma imply that the Lyapunov function provides a bound on how the state converges as a function of iterate and the ultimate bound of the state.

VII.A. Linear Designs

For the zero-input general linear system as defined by Eq. (6) the following theorem yields information regarding the exponential bounds of the design (*i.e.*, how fast the design converges)[18]

Theorem 3 (Linear System Exponential Stability)**.** *The origin of Eq. (6) with* $\mathbf{u}_k = \mathbf{0}\ \forall k \in \mathbb{Z}_+$ *is uniformly (exponentially) asymptotically stable if, and only if, there exists a sequence of nonsingular matrices* $\mathbf{W}_k \in \mathbb{C}^{n \times n}$ *and some matrix norm* $\parallel \cdot \parallel$*, with* $\parallel \mathbf{W}_k \parallel$ *and* $\parallel \mathbf{W}_k^{-1} \parallel$ *uniformly bounded, and* $\beta \triangleq \sup_k \{\beta_k\} < 1$ *where* $\beta_k \triangleq \parallel \mathbf{W}_{k+1}\mathbf{A}_k\mathbf{W}_k^{-1} \parallel$*. In this case, given any initial state* $\mathbf{x}_0 \in \mathbb{R}^n$ *and defining* $w \triangleq \sup_k \parallel \mathbf{W}_k^{-1} \parallel$*,* $\parallel \mathbf{x}_k \parallel \leq \beta^k w \parallel \mathbf{W}_0 \mathbf{x}_0 \parallel$

Proof of this theorem is found in Ref. 18. This theorem says that if the linear system describing the convergence of the design is transformed according to

$$\boldsymbol{\zeta}_k = \mathbf{W}_k \mathbf{x}_k \tag{21}$$

then

$$\boldsymbol{\zeta}_{k+1} = \boldsymbol{\Psi}_k \boldsymbol{\zeta}_k \tag{22}$$

where $\boldsymbol{\Psi}_k \triangleq \mathbf{W}_{k+1}\mathbf{A}_k\mathbf{W}_k^{-1}$. Due to the condition $\beta_k < 1$, $\parallel \boldsymbol{\Psi}_k \parallel < 1$ and the transformed system is a contraction mapping.

The computation of the matrix $\mathbf{W}_k$ for the case when $\mathbf{A}_k = \mathbf{A}\ \forall k \in \mathbb{Z}_+$ is significantly more tractable and can be readily achieved by any of the following methods[18]

1. If $\mathbf{A}$ is diagonalizable, $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ where $\mathbf{D} \triangleq \text{diag}\{\lambda_i\}$, then choosing $\mathbf{W} = \mathbf{V}^{-1}$ and $\parallel \cdot \parallel_2$ gives $\beta = \lambda_{\max}$.
2. For any $\mathbf{A}$, compute $\mathbf{A} = \mathbf{U}^*\mathbf{R}\mathbf{U}$, the Schur decomposition and set $\mathbf{W} = \boldsymbol{\Gamma}\mathbf{U}$ where $\boldsymbol{\Gamma} = \text{diag}\{1, \gamma, \gamma^2, \cdots, \gamma^{n-1}\}$.
3. Choose a positive definite $\mathbf{Q}$ and solve $\mathbf{P} - \mathbf{A}^T\mathbf{P}\mathbf{A} = \mathbf{Q}$ to obtain a positive definite $\mathbf{P}$. Compute the Cholesky factorization $\mathbf{P} = \mathbf{W}^T\mathbf{W}$.

Each of these provide a value of β which can be used as an absolute scale to describe how fast a design will converge as the norm of the of the state $\parallel \mathbf{x} \parallel$ decreases by a factor proportional to β at each iterate.

VII.B. Nonlinear Designs

The methods of linear systems can be extended to nonlinear designs, that is those designs whose iteration is described by Eq. (5). The following theorem provides a sufficient condition for exponential stability, a domain of exponential stability, and exponential bounds on the state.[18]

Theorem 4 (Nonlinear System Exponential Stability)**.** *The origin of Eq. (5) with* $\mathbf{u}_k = \mathbf{0}\ \forall k \in \mathbb{Z}_+$ *is exponentially asymptotically stable if there exists a nonsingular matrix* $\mathbf{W} \in \mathbb{C}^{n\times n}$ *and some matrix norm* $\|\cdot\|$*, such that*

$$\beta \triangleq \sup_k \sup_{\mathbf{v}\in\Omega} \left\| \mathbf{W} \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{v}) \right] \mathbf{W}^{-1} \right\| < 1$$

for some open convex set $\Omega \in \mathbb{R}^n$ *with* $\mathbf{0} \in \Omega$*. There exists an open set* $\mathcal{X}_s \subseteq \Omega$ *with* $\mathbf{0} \in \mathcal{X}_s$*, and* $\forall \mathbf{x}_0 \in \mathcal{X}_s$*,* $\exists \beta_0 \in [0, \beta]$ *such that* $\| \mathbf{x}_k \| \leq \beta_0^k \kappa(\mathbf{W}) \| \mathbf{x}_0 \|$*, and hence* $\mathcal{X}_s$ *is a domain of exponential stability.*

This is proven in Ref. 18. However, this theorem again provides a rate of convergence, provided the associated conditions are met. In this case, the rate of convergence is given as β_0 as the magnitude of the initial state is reduced successively by this amount.

VIII. Examples

VIII.A. A Linear, Three Contributing Analysis System

Consider the linear, three CA system shown in Fig. 4 where each CA is scalar. In this case, it is desired to

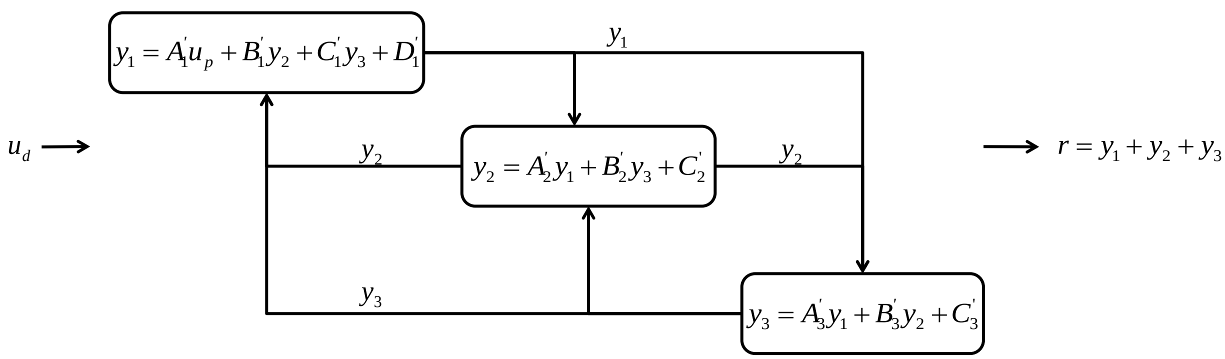


Figure 4. Three contributing analysis multidisciplinary design.

find $u_d \in \mathbb{R}$ that minimizes the summation of the CAs output while being within the unit cube centered at the origin. In other words

$$\left.\begin{array}{ll} \textbf{Minimize}: & \mathcal{J} = y_1 + y_2 + y_3 \\ \textbf{Subject to}: & y_1, y_2, y_3 \in [-1, 1] \\ \textbf{By varying}: & u_d \end{array}\right\}$$

The output of the CAs can be stated in the form of a linear system

$$y_1 = \begin{pmatrix} 0 & B_1' & C_1' \end{pmatrix} \mathbf{y} + \begin{pmatrix} A_1' \end{pmatrix} u_d + \begin{pmatrix} \mathbf{0} \end{pmatrix} \mathbf{u_p} + D_1'$$

Similarly, for the second CA, the functional form is given by

$$y_2 = \begin{pmatrix} A_2' & 0 & B_2' \end{pmatrix} \mathbf{y} + \begin{pmatrix} 0 \end{pmatrix} u_d + \begin{pmatrix} \mathbf{0} \end{pmatrix} \mathbf{u_p} + C_2'$$

and the third CA

$$y_3 = \begin{pmatrix} A_3' & B_3' & 0 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 0 \end{pmatrix} u_d + \begin{pmatrix} \mathbf{0} \end{pmatrix} \mathbf{u_p} + C_3'$$

Hence,

$$\mathbf{A_1} = \begin{pmatrix} 0 & B_1' & C_1' \end{pmatrix} \quad \mathbf{B_1} = \begin{pmatrix} A_1' \end{pmatrix} \quad \mathbf{C_1} = \begin{pmatrix} \mathbf{0} \end{pmatrix} \quad \mathbf{d_1} = D_1'$$

$$\mathbf{A_2} = \begin{pmatrix} A_2' & 0 & B_2' \end{pmatrix} \quad \mathbf{B_2} = \begin{pmatrix} 0 \end{pmatrix} \quad \mathbf{C_2} = \begin{pmatrix} \mathbf{0} \end{pmatrix} \quad \mathbf{d_2} = C_2'$$

$$\mathbf{A_3} = \begin{pmatrix} A_3' & B_3' & 0 \end{pmatrix} \quad \mathbf{B_3} = \begin{pmatrix} 0 \end{pmatrix} \quad \mathbf{C_3} = \begin{pmatrix} \mathbf{0} \end{pmatrix} \quad \mathbf{d_3} = C_3'$$

The fixed-point iteration use to converge the design is defined by the relation

$$\mathbf{y}_k = \mathbf{f}(\mathbf{y}_{k-1}), \quad \forall k \in \mathbb{Z}_+ \setminus \{0\} \tag{23}$$

where $\mathbf{f}(\mathbf{y}_{k-1})$ is the output value of the CAs on the $k^{\text{th}} - 1$ iteration. In terms of the linear equations developed the fixed-point iteration equations are

$$\mathbf{y}_k = \mathbf{\Lambda}\mathbf{y}_{k-1} + \boldsymbol{\beta}\mathbf{u_d} + \boldsymbol{\gamma}\mathbf{u_p} + \boldsymbol{\delta} \tag{24}$$

For this example, the fixed-point iteration equations described in Eq. (24) are

$$\mathbf{\Lambda} = \begin{pmatrix} \mathbf{A_1} \\ \mathbf{A_2} \\ \mathbf{A_3} \end{pmatrix} = \begin{pmatrix} 0 & B_1' & C_1' \\ A_2' & 0 & B_2' \\ A_3' & B_3' & 0 \end{pmatrix}$$

$$\boldsymbol{\beta} = \begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \\ \mathbf{B_3} \end{pmatrix} = \begin{pmatrix} A_1' \\ 0 \\ 0 \end{pmatrix}$$

$$\gamma = \begin{pmatrix} \mathbf{C_1} \\ \mathbf{C_2} \\ \mathbf{C_3} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\delta} = \begin{pmatrix} \mathbf{d_1} \\ \mathbf{d_2} \\ \mathbf{d_2} \end{pmatrix} = \begin{pmatrix} D_1' \\ C_2' \\ C_3' \end{pmatrix}$$

VIII.A.1. Design Results

The parameters used within the models for each of the cases examined are shown in Table 2 where the values without distributions are assumed to be deterministic.

Table 2. Parameters for the design of a three contributing analysis system.

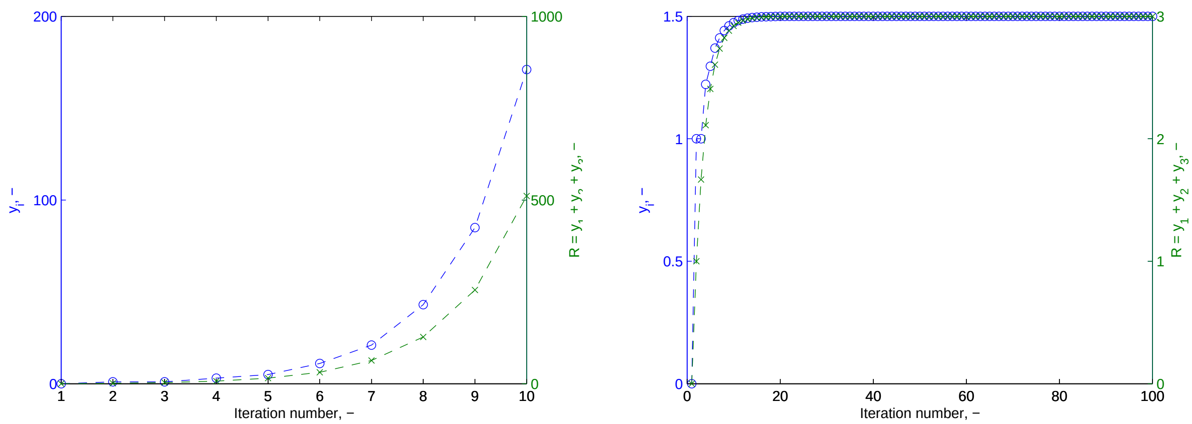
Parameter	Case 1	Case 2
$\mathbf{\Lambda}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 0 \end{pmatrix}$
$\boldsymbol{\beta}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
$\boldsymbol{\gamma}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
$\boldsymbol{\delta}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Case 1: Divergent Design

In the first case, the nominal eigenvalues of $\mathbf{\Lambda}$ are found to be

$$\lambda = \{-1, -1, 2\}$$

Hence, since $|\lambda_{\max}| \geq 1 \; \forall \lambda_i, \; i \in \{1, 2, 3\}$ there is not a feasible design to be found with the iteration scheme. This is shown in Fig. 5(a) where the objective function value exponentially diverges. Despite fixed-point



(a) (b)

Figure 5. Fixed-point iteration values for a (a) convergent design and (b) divergent design

iteration not being able to find a solution, a feasible design does exist. The feasible designs are characterized by the equation

$$\mathbf{y} = \begin{pmatrix} 0 \\ -1/2 \\ -1/2 \end{pmatrix} u_d$$

which implies that the optimum is found with $u_d = 2$. This example demonstrates the need for alternative iteration schemes to be investigated.

Case 2: Convergent Design

In the second case, the nominal eigenvalues of $\mathbf{\Lambda}$ (see Table 2) are substantially different,

$$\lambda = \{-1/3, -1/3, 2/3\}$$

This implies that a feasible solution should be able to be found since $|\lambda_{\max}| = 2/3 \leq 1$. This fact is demonstrated in Fig. 5(b) where the objective function value converges for an arbitrary value of u_d. The optimal design in his case is found to be when $u_d = -2/3$, which has an objective value $r^* = -2$.

VIII.B. A Nonlinear, Two Contributing Analysis System

Consider the multidisciplinary design with fixed-point iteration defined by

$$\left.\begin{aligned} y_{1,k+1} &= \frac{1}{2} y_{1,k} + \frac{1}{2} y_{2,k} \\ y_{2,k+1} &= \alpha y_{1,k}^3 + \frac{1}{4} y_{2,k} \end{aligned}\right\} \tag{25}$$

With

$$\mathbf{W} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$$

and defining $a \triangleq 24\alpha y_1^2$ it is shown that $\beta < 1$ for $a \in [-1.9, 1.4]$, which shows the origin is exponentially asymptotically stable for any finite α. With $\alpha = -0.1$ the domain of attraction can be shown to be

$\mathcal{A} = \{\mathbf{y} | -0.89 < y_1 < 0.89\}$ and with $V(\mathbf{y}) = \| \mathbf{W}\mathbf{y} \|_2$ it can be shown that $\mathcal{X}_s = \{\mathbf{y} | V(\mathbf{y}) < 0.63\} \subset \mathcal{A}$. If $\| \mathbf{y}_0 \|_2 < 0.2$ then $\mathbf{y}_0 \in \mathcal{X}_s$ then $\beta_0 = 0.56$ allows the bound on convergence to be shown as $\| \mathbf{y}_k \|_2 < 0.9(0.56)^k$. Note that other choices of norms or $\mathbf{W}$ could yield different values.

IX. Conclusions

This work has shown the applicability of dynamical system theory to the convergence of multidisciplinary designs. In particular, the stability of the dynamical systems describing the convergence of the design was examined with specific criterion given in terms of general method using Lyapunov theory and more tractable methods for specific functional cases. Asymptotic stability of the dynamical system implies that the multidisciplinary design will converge. For linear systems, the instability of the system implies that the system will not converge for the choice of root-finding schemes. A method to estimate the domain of attraction, that is the range of initial guesses which will cause the design to converge, was discussed as was an innovative method to find Lyapunov functions for discrete dynamical systems. Finally, the rate of convergence for the iteration was discussed for exponentially stable iteration schemes with specific criterion given for linear and nonlinear designs.

Appendix

This appendix contains the proofs for the theorems presented throughout this work.

Lyapunov's Direct Method

The following proof for Lyapunov's direct method follows that outlined in Refs. 10 and 11.

Proof: Lyapunov's Direct Method for Discrete Dynamical Systems. Choose $r_0 > 0$ such that $\{\mathbf{x}_k : \| \mathbf{x}_k - \mathbf{x}_e \| \leq r_0\} \subset \mathcal{S} \cap \mathcal{D}$. By the continuity of $\mathbf{f}$ there is an $r_1 \leq r_0$ such that $\| \mathbf{f}(\mathbf{x}_k) - \mathbf{x}_e \| \leq r_0$ whenever $\| \mathbf{x}_k - \mathbf{x}_e \| \leq r_1$. Now let $\epsilon > 0$ be given and assume, without loss of generality, that $\epsilon \leq r_1$. Then choose $\delta \in (0, \epsilon)$ so that $\| \mathbf{x}_k - \mathbf{x}_e \| \leq \delta$ implies that

$$V(\mathbf{x}_k) < \phi(\epsilon) \equiv \min\{V(\mathbf{x}_k) : \epsilon \leq \| \mathbf{x}_k - \mathbf{x}_e \| \leq r_0\}$$

This can be achieved using the continuity of V and the fact that $V(\mathbf{x}_k)$ is positive definite. Now suppose there is some $\mathbf{x}_0$ such that $\| \mathbf{x}_0 - \mathbf{x}_e \| \leq \delta$ but $\| \mathbf{x}_{k+1} - \mathbf{x}_e \| > \epsilon$ for some k. Assume that this is the first such k; thus $\| \mathbf{x}_i - \mathbf{x}_e \| \leq \epsilon \leq r_1, \ i = 1, 2, \cdots, k$. Then $\| \mathbf{f}(\mathbf{x}_k) - \mathbf{x}_e \| \leq r_0$ so that $V(\mathbf{f}(\mathbf{x}_k))$ is well-defined and $V(\mathbf{f}(\mathbf{x}_k)) \geq \phi(\epsilon)$. But by the definition of a Lyapunov function

$$V(\mathbf{x}_{k+1}) \leq V(\mathbf{x}_k) \leq \cdots \leq V(\mathbf{x}_0) < \phi(\epsilon)$$

This is a contradiction and stability is proved.

For asymptotic stability, it suffices to consider any sequence $\{\mathbf{x}_k\} \subset \{\mathbf{x}_k : \| \mathbf{x}_k - \mathbf{x}_e \| \leq \epsilon\}$ and show that $\mathbf{x}_k \to \mathbf{x}_e$ as $k \to \infty$, and for this it suffices to show that if $\hat{\mathbf{x}}$ is any limit point of $\{\mathbf{x}_k\}$, then $\hat{\mathbf{x}} = \mathbf{x}_e$. Suppose not, then the mapping

$$r(\mathbf{x}_k) = \frac{V(\mathbf{f}(\mathbf{x}_k))}{V(\mathbf{x}_k)}$$

is well-defined and continuous in some open neighborhood $\mathcal{S}_0$ of $\hat{\mathbf{x}}$ and since $\Delta V < 0$, $r(\hat{\mathbf{x}}) < 1$. Hence, for a given $\alpha \in (r(\hat{\mathbf{x}}), 1)$, there is a $\delta > 0$ such that $r(\mathbf{x}_k) \leq \alpha$ whenever $\| \mathbf{x}_k - \hat{\mathbf{x}} \| \leq \delta$. Therefore, for sufficiently large k_i, the subsequence converging to $\hat{\mathbf{x}}$ satisfies

$$V(\mathbf{x}_{k_i+1}) = V(\mathbf{f}(\mathbf{x}_{k_i+1})) \leq \alpha V(\mathbf{x}_{k_i}) \leq \cdots \leq \alpha V(\mathbf{x}_{k_{i-1}+1}) \leq \cdots \leq \alpha^i V(\mathbf{x}_0)$$

so that $V(\mathbf{x}_{k_i}) \to 0$ as $i \to \infty$. But the continuity of V implies that $V(\hat{\mathbf{x}}) = 0$, and because the Lyapunov function is positive definite, $\hat{\mathbf{x}} = \mathbf{x}_e$ proving asymptotic stability.

For global asymptotic stability, note that for any $\mathbf{x}_0$, the radial unboundedness of the Lyapunov function guarantees that $\{\mathbf{x}_k\}$ is bounded otherwise there would be a subsequence $\{\mathbf{x}_{k_i}\}$ such that $\| \mathbf{x}_{k_i} - \mathbf{x}_e \| \to \infty$ as $i \to \infty$ and hence $V(\mathbf{x}_{k_i}) \to \infty$ as $i \to \infty$. This contradicts the monotone decreasing behavior of $V(\mathbf{x}_k)$ required by $\Delta V < 0$. It now follows precisely as in the case of asymptotic stability that $\mathbf{x}_k \to \mathbf{x}_e$ as $k \to \infty$ proving global asymptotic stability. □

Region of Attraction

This proof follows that outlined in Ref. 13

Proof: Region of Attraction. If $\mathbf{x}_0 \in \mathcal{A}_0$ then the third condition on $w(\mathbf{x})$ shows that $w(\mathbf{x}_1) \le w(\mathbf{x}_0)$ so that $\mathbf{x}_1 \in \mathcal{A}_0$ and, by induction, $\mathbf{x}_k \in \mathcal{A}_0$ and $w(\mathbf{x}_{k+1}) \le w(\mathbf{x}_k),\ k = 2, \cdots$. Hence the sequence $\{w(\mathbf{x}_k)\}$ converges. Also note that the third condition on $w(\mathbf{x})$ implies that

$$\frac{1 - w(\mathbf{f}(\mathbf{x}))}{1 - w(\mathbf{x})} = 1 + \phi(\mathbf{x}), \quad \mathbf{x} \in \mathcal{A}_0$$

so that

$$\frac{1 - w(\mathbf{x}_k)}{1 - w(\mathbf{x}_0)} = \prod_{i=0}^{k-1} \frac{1 - w(\mathbf{x}_{i+1})}{1 - w(\mathbf{x}_i)} = \prod_{i=0}^{k-1} 1 + \phi(\mathbf{x}_i)$$

Since the left hand side of this equality converges as $k \to \infty$, the right does as well which implies that $\phi(\mathbf{x}_k) \to 0$ as $k \to \infty$. Then the first condition on $w(\mathbf{x})$ and the continuity of ϕ ensures that $\mathbf{x}_k \to \mathbf{x}_e$ as $k \to \infty$. Conversely, suppose that $\mathbf{x}_0 \notin \mathcal{A}_0$ then the third condition on $w(\mathbf{x})$ shows that $w(\mathbf{x}_1) \ge w(\mathbf{x}_0) \ge 1$ so that $\mathbf{x}_1 \notin \mathcal{A}_0$ and, by induction, $w(\mathbf{x}_k) \ge 1,\ k = 2, 3, \cdots$. But if $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}_e$, then the continuity of w requires that $\lim_{k\to\infty} w(\mathbf{x}_k) = 0$ which is a contradiction. □

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