AAS 13-794

FORMATION CONTROL PROBLEMS FOR DECENTRALIZED SPACECRAFT SYSTEMS

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This paper investigates controlling spacecraft formations, as opposed to individual spacecraft in formation, with the intent to determine whether Leader-Follower formations are error- or fuel-optimal. An formation optimal control problem is developed wherein formation slots and spacecraft barycenter weights are the formation control variables. Restricted formation control problems are presented which investigate the effect of controlling the individual spacecraft barycenter weights based on minimum Lyapunov formation error and also based on minimum fuel costs. It is shown that a Democratic formation works well with no *a-priori* knowledge of the formation state or orbit elements. Further, detailed simulation results suggest that Democratic formations tend to have lower overall fuel usage than strict Leader-Follower formations. Conclusions and future work are discussed.

INTRODUCTION

There is an ever growing need for solutions to spacecraft formation control problems due to increasing complexity in spacecraft systems. The fractionation approach to spacecraft systems involves decomposing large spacecraft into smaller, specialized spacecraft which work together to achieve shared goals.[1] There are advantages to this method, such as improved reliability and allowing a system to increase its coverage area at a lower overall cost;[2] however, this drastically increases the complexity of the system and its on-board autonomy systems. Due to restrictions in uplink/downlink connectivity it is neither feasible for a central ground control system to control the entire formation, nor is it always practical for a single spacecraft to bear the burden of centralized Guidance, Navigation, and Control (GNC) for the entire system. Therefore, a decentralized formation GNC system is desirable.

Orbiting bodies are subject to oscillating, gross, and differential perturbations. In the presence of perturbation, mean orbit elements essentially 'average out' small oscillations in each spacecraft's orbit, and have a one-to-one mapping with instantaneous orbit elements.[3,4] Gross perturbations affect the entire formation and can incur large $\Delta \mathbf{v}$ costs if rejected. In formation control, it is preferable to ignore gross perturbations in the short term and focus primarily on relative orbit maintenance. Further, unperturbed differential mean orbit elements, with respect to a formation barycenter, behave similarly to constants of motion for arbitrary orbit regimes, and change slowly under perturbations. For these reasons, they are ideal for use as consensus variables in formation flight systems.

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Current formation control formulations typically assume a Leader/Follower (L/F) architecture.[5,6] This architecture is advantageous because it reduces the formation control problem to a tracking problem where each follower spacecraft has an error directly relative to the leader. The issue with this architecture is that it defines the leader spacecraft as having zero error and therefore uses no fuel, which may or may not be desired. A way of mitigating this problem is to switch the leadership designations between spacecraft in the formation to balance fuel.[7] However, it is possible that Leader/Follower formations are not the ideal solution for optimizing fuel balancing, or for minimizing fuel usage of the formation as a whole. This paper investigates formation design and control to identify minimum Δv formations.

Previous works by the authors have outlined methods for defining a weighted formation barycenter for the spacecraft formation. It has been shown that by using differential mean orbit elements as consensus variables, even with intermittent communication links between each spacecraft in a formation, the system is able to reach consensus on formation state estimates.[8] It has also been shown that with this system definition, the spacecraft can be controlled in continuous time to eliminate their state error in the presence of navigation error.[9] Further building on decentralized mean orbit element formulation, sufficient conditions for formation stability in the presence of uncoordinated impulsive maneuvers have been shown to exist.[10]

With the groundwork of individual spacecraft formation control slots and trajectory maintenance in place, it is possible to look at problems involving the formation as a whole. A formation definition which conveniently separates the formation control problem from the individual spacecraft control problem is needed, which will allow the formation control problem framework to be applied to any spacecraft formation. Formation control is different from individual spacecraft control. The spacecraft may have arbitrary control policies, and the formation control definition still needs to be able to manage these. Using this definition, questions about what formations are best for the spacecraft system as a whole can be investigated.

The main contributions of this paper are to 1) develop a dynamical system definition for a general spacecraft formation that uses differential mean orbit elements and allows for varying spacecraft weights; 2) propose an optimal formation control problem for controlling individual spacecraft weights and mean orbit element formation slots, 3) find a minimum formation Lyapunov error formation design, and, finally, 4) to find a minimum Δv feedback formation controller for formation weights.

This paper begins by reviewing necessary concepts developed in previous works. Next, a full nonlinear optimal formation control problem is be presented. Due to the complexity of the optimal formation control problem, more restricted formation control problems are proposed and addressed. These include a formulation for a minimum formation Lyapunov error weighting scheme as well as a minimum Δv feedback formation controller for weights. A simulation is developed and several test cases for different control laws are presented and compared to evaluate the efficacy of the minimum Δv feedback formation controller. Lastly, conclusions and future work is discussed.

PREVIOUS DISTRIBUTED FORMATION FLIGHT WORK

To provide context for the contributions of this paper, definitions from previous works by Holzinger & McMahon[8,10] which describe the distributed spacecraft formation must be introduced. This paper uses the classical orbital elements, a, e, i, Ω, ω, and M.

Definition 1. ***Weighted Formation Barycenter***[8]

The formation mean orbit element barycenter $\bar{\mathbf{œ}}_k^b$ at time t_k given weights w_k^i and reference differential mean orbit elements $\delta\bar{\mathbf{œ}}_{r,k}^i$ is defined as

$$\bar{\mathbf{œ}}_k^b = \sum_{i=1}^{N_f} w_k^i \bar{\mathbf{œ}}_k^{b,i} = \sum_{i=1}^{N_f} w_k^i (\bar{\mathbf{œ}}_k^i - \delta\bar{\mathbf{œ}}_{r,k}^i) \tag{1}$$

where the weights $w_k^i \in \mathbb{R}$ are such that $0 \leq w_k^i \leq 1, i = 1, \ldots, N_f$, and $w_k^1 + \cdots + w_k^{N_f} = 1$.

The weighted formation barycenter is the weighted average of each spacecraft's calculated barycenter, $\bar{\mathbf{œ}}_k^{b,i}$. The calculated barycenter is each individual spacecraft's mean orbit elements offset by its differential mean orbit element reference offset (or formation 'slot'), $\delta\bar{\mathbf{œ}}_{r,k}^i$.

Definition 2. ***Formation Slot***[8]

A formation slot is defined by a spacecraft's choice of $\delta\bar{\mathbf{œ}}_{r,k}^i$. The only constraints placed on $\delta\bar{\mathbf{œ}}_{r,k}^i$ is that they be well defined on the orbit element space (e.g., differential mean eccentricity $\bar{e}_k^i$ such that $0 \leq \bar{e}_k^b + \delta\bar{e}_k^i < 1$) and satisfy user-defined constraints, such as collision avoidance and other operational needs.

Definition 3. ***Relative Formation***[8]

A spacecraft formation is said to be a relative formation when each spacecraft is aware of all other formation spacecraft (i.e. given a set of formation spacecraft $\mathcal{F}$, the size of $\mathcal{F}$, N_f, is known, and a specific spacecraft is associated with each $i \in \mathcal{F}$), their respective current mean orbit elements $\bar{\mathbf{œ}}_k^i$, and barycenter weightings w_k^i. Further, each spacecraft must know its own differential mean orbit element formation slot $\delta\bar{\mathbf{œ}}_{r,k}^i$.

The value of the differential mean orbit elements is given by

$$\delta\bar{\mathbf{œ}}_k^i = \bar{\mathbf{œ}}_k^i - \bar{\mathbf{œ}}_k^b \tag{2}$$

which can then be used to define the instantaneous differential mean orbit element error:

$$\delta\bar{\mathbf{e}}_k^i = \bar{\mathbf{œ}}_k^i - \bar{\mathbf{œ}}_{r,k}^i \tag{3}$$

When $\delta\bar{\mathbf{e}}_k^i \to \mathbf{0} \; \forall i \in \mathcal{F}$, the formation has zero error. With the definitions pertaining to the spacecraft formation in place, Definitions, Lemmas, and Corollaries necessary to describe formation stability can be introduced. Often the time subscript $(\cdot)_k$ is dropped as a matter of convenience and should be considered to be implicit. To begin the stability relations, the Lyapunov function used to measure stability must be introduced.

Definition 4. ***Formation Error Lyapunov Function***[10]

The formation error Lyapunov function is defined as

$$V = \frac{1}{2} \sum_{i \in \mathcal{F}} (\delta\bar{\mathbf{e}}^i \cdot \delta\bar{\mathbf{e}}^i) \tag{4}$$

The mean orbit element state error for each spacecraft i in the formation $\mathcal{F}$ is weighted equally. Note the definition of V in (4) is such that $V \geq 0 \;\; \forall \, \delta\bar{\mathbf{e}}^i, \, i \in \mathcal{F}$.

Now that the Lyapunov function has been defined, a condition for Lyapunov-stable impulsive spacecraft maneuvers derived, in Reference 10, is given in Lemma 1.

Lemma 1. ***Impulsive Formation Control Stability Sufficient Condition***

Given a formation $\mathcal{F}$ with N_f spacecraft, a set of N_m simultaneously maneuvering spacecraft $\mathcal{M} \subseteq \mathcal{F}$, where $\mathcal{M} \neq \emptyset$, and a set of N_q quiescent spacecraft $\mathcal{Q} \subset \mathcal{F}$, where $\mathcal{M} \cup \mathcal{Q} = \mathcal{F}$, and $\mathcal{M}$ and $\mathcal{Q}$ are disjoint, where the estimated mean orbit element states of the formation are all true ($\hat{\bar{\text{œ}}}^i \rightarrow \bar{\text{œ}}^i$) and all weights w^j and differential mean orbit elements $\delta\bar{\text{œ}}^j$ are known and agreed upon by the formation, a sufficient condition for formation stability under impulsive maneuvers resulting in a mean orbit element change $\bar{\text{œ}}^{i+} = \bar{\text{œ}}^i + \Delta\bar{\text{œ}}^i$ for each satellite i maneuvering impulsively at any time-step k is

$$\sum_{i \in \mathcal{M}} \left\{ \frac{1}{2} k(w^i, N_f) \Delta\bar{\text{œ}}^i \cdot \Delta\bar{\text{œ}}^i - \left(\delta\bar{\mathbf{e}}^i - w^i \sum_{j \in \mathcal{F}} \delta\bar{\mathbf{e}}^j \right) \cdot \Delta\bar{\text{œ}}^i + w^i \left[\sum_{j \neq i \in \mathcal{M}} c(w^j, N_f) \Delta\bar{\text{œ}}^j \right] \cdot \Delta\bar{\text{œ}}^i \right\} \leq 0 \tag{5}$$

where

$$k(w^i, N_f) = 1 - 2w^i + N_f (w^i)^2 \tag{6}$$

and

$$c(w^j, N_F) = \frac{N_F}{2} w^j - 1 \tag{7}$$

Proof. *See Ref. 10.* □

Since spacecraft perform maneuvers in a 3-dimensional Inertial space, rather than in orbital element space, a transfer function which projects the movement in Inertial space to orbital element space must be used. Therefore, only a 3-dimensional subspace of the orbital elements can actually be controlled. With this in mind, a spacecraft's impulsive maneuver can be described in oscillating orbit elements as

$$\begin{aligned} \text{œ}^{i+} &= \text{œ}^i + \Delta\text{œ}^i \\ &= \text{œ}^i + \mathbf{B}(\text{œ}^i)\Delta\mathbf{v}^i \end{aligned} \tag{8}$$

where $\Delta\mathbf{v}$ is in a rotating Hill frame. $\mathbf{B}(\cdot)$ represents Gauss' variational equations. The $\mathbf{B}$ matrix is defined as

$$\mathbf{B}(\text{œ}^i) = \begin{bmatrix} \frac{2a^2 e \sin f}{h} & \frac{2a^2 p}{hr} & 0 \\ \frac{p \sin f}{h} & \frac{(p+r)\cos f + re}{h} & 0 \\ 0 & 0 & \frac{r \cos\theta}{h} \\ 0 & 0 & \frac{r \sin\theta}{h \sin i} \\ -\frac{p \cos f}{he} & \frac{(p+r)\sin f}{he} & -\frac{r \sin\theta \cos i}{h \sin i} \\ \frac{\eta(p\cos f - 2re)}{he} & -\frac{\eta(p+r)\sin f}{he} & 0 \end{bmatrix} \tag{9}$$

where $\eta = \sqrt{1-e^2}$, p is the semilatus rectum, h is the angular momentum of the orbit, r is the orbit radius, and $\theta = \omega + f$ is the true latitude angle. Note that Equation (8) is in terms of oscillating orbital elements. It can be assumed that as long as the impulses are small enough, the difference between impulses in oscillating and mean orbital elements is negligible,[3] leading to the following simplification,

$$\Delta\bar{\text{œ}}^i \approx \Delta\text{œ}^i = \mathbf{B}(\text{œ}^i)\Delta\mathbf{v}^i \tag{10}$$

The following definitions introduce special cases which are useful for describing formations.[10]

Definition 5. ***Leader / Follower Formation***

A formation $\mathcal{F}$ is a Leader/Follower (L/F) formation if and only if $w^i = 1$ (designating the i^{th} spacecraft as the leader) and all other $j \neq i \in \mathcal{F}$ have weightings such that $w^j = 0$.

Definition 6. ***Democratic Formation***

A formation $\mathcal{F}$ is a democratic formation if and only if for all $i \in \mathcal{F}$, $w^i = 1/N_f$, where N_f is the number of spacecraft in $\mathcal{F}$.

For the remainder of this paper, it is assumed that each spacecraft in the formation will perform asynchronous maneuvers. Meaning that no two spacecraft will perform a maneuver at the same time. This eliminates any complexity involved in spacecraft performing simultaneous maneuvers, as well as the necessity for spacecraft to coordinate maneuvers with each other. Under this assumption, Eqn. (5) can be simplified to

$$\frac{1}{2}k(w^i, N_f)\Delta\bar{\text{œ}}^i \cdot \Delta\bar{\text{œ}}^i - \left(\delta\bar{\mathbf{e}}^i - w^i \sum_{j\in\mathcal{F}} \delta\bar{\mathbf{e}}^j\right) \cdot \Delta\bar{\text{œ}}^i \leq 0 \tag{11}$$

since each maneuver only involves one spacecraft. It can then be found that the impulsive maneuver which maximally decreases the formation state error is

$$\Delta\bar{\text{œ}}^{i,*} = \frac{1}{1 - 2w^i + N_f(w^i)^2}\left(\delta\bar{\mathbf{e}}^i - w^i\left[\sum_{j\in\mathcal{F}} \delta\bar{\mathbf{e}}^j\right]\right) \tag{12}$$

This minimum Lyapunov error $\Delta\bar{\text{œ}}^i$ can be simplified for certain special cases. For a 'follower' spacecraft i, as $w^i \to 0$,

$$\Delta\bar{\text{œ}}^{i,*}\,|_{w^i\to 0} = \delta\bar{\mathbf{e}}^i$$

where the optimal maneuver is simply to remove spacecraft i's error. This occurs because when a spacecraft's weighting is equal to zero, it cannot affect the error of any other spacecraft in the formation. For a 'leader' spacecraft, as $w^i \to 1$,

$$\Delta\bar{\text{œ}}^{i,*}\,|_{w^i\to 1} = -\frac{1}{N_f - 1}\left[\sum_{j=\neq i\in\mathcal{F}} \delta\bar{\mathbf{e}}^j\right]$$

Note that when a spacecraft is set as the leader, it ignores its own error (as it has none since $\delta\bar{\mathbf{e}}^i \to 0$ as $w^i \to 1$) and instead maximally decreases the weighted formation state error. For a democratic formation, as $w^i \to 1/N_f$,

$$\Delta\bar{\text{œ}}^{i,*}\,|_{w^i\to\frac{1}{N_f}} = \frac{N_f}{N_F - 1}\left(\delta\bar{\mathbf{e}}^i - \frac{1}{N_f}\left[\sum_{j\in\mathcal{F}} \delta\bar{\mathbf{e}}^j\right]\right)$$

Under the assumptions of Lemma 1 and asynchronous maneuvers, any maneuver which can be characterized as

$$\Delta\bar{\text{œ}}^i = \Delta\bar{\text{œ}}^{i,*} + \varepsilon^i \|\Delta\bar{\text{œ}}^{i,*}\|\hat{\mathbf{r}}^i \tag{13}$$

where $\varepsilon \in [0,1]$ and $\hat{\mathbf{r}}^i$ is any 6-dimensional unit vector, will satisfy Lemma 1 and is thereby a formation stabilizing maneuver with respect to the Lyapunov function 4. Equation 13 describes a 6-dimensional sphere in which formation stable $\Delta\bar{\text{œ}}^i$ maneuvers exist. Note that null maneuvers ($\Delta\bar{\text{œ}}^i = \mathbf{0}$) are also admissible. This result implies that there is a 3-dimensional volume, which is not necessarily a sphere, containing admissible $\Delta\mathbf{v}^i$ maneuvers that are stabilizing. Using the transformation in Eqn. (10), an equation for the minimum Lyapunov $\Delta\mathbf{v}$ maneuver can be found

$$\Delta\mathbf{v}^{i,*} = \mathbf{B}(\text{œ}^i)^+ \Delta\bar{\text{œ}}^{i,*} \tag{14}$$

It is generally accepted that a Leader/Follower formation is the default formation for formation flight. Now that a framework for describing a distributed formation flight is in place, the question of whether or not a Leader/Follower formation is actually the most efficient can be investigated. The formation flight framework introduced in the previous section includes the weighting scheme as a function of time, allowing the effect of varying the weights and relative formation slots over time to be investigated. The following section defines a general optimal control problem for a distributed spacecraft formation.

Optimal Formation Control Problem

The optimal control problem is formulated using a state vector, $\mathbf{x}_k = \mathbf{x}(t_0 + k\Delta t)$, which is a concatenation of the individual spacecraft weights, w_k^i, formation slots, $\delta\bar{\text{œ}}_{r,k}^i$, and $\bar{\text{œ}}_k^i$. The system is controlled by the decision variable $\mathbf{u}_k = \mathbf{u}(t_0 + k\Delta t)$, which is the change in state between two time steps. The state and control variable are then shown by

$$\mathbf{x}_k = \left[\ \cdots\ w_k^i\ \cdots\ |\ \cdots\ \left[\delta\bar{\text{œ}}_{r,k}^i\right]^T\ \cdots\ |\ \cdots\ \bar{\text{œ}}_k^i\ \cdots\ \right]^T \tag{15}$$

$$\mathbf{u}_k = \left[\ \cdots\ \Delta w_k^i\ \cdots\ |\ \cdots\ \left[\Delta\delta\bar{\text{œ}}_{r,k}^i\right]^T\ \cdots\ \right]^T \tag{16}$$

where w_k^i is the ith spacecraft's individual weighting, $\delta\bar{\text{œ}}_{r,k}^i$ is its formation slot, $\bar{\text{œ}}_k^b$, Δw_k^i is the ith spacecraft's instantaneous change in weighting at each time step, and $\Delta\delta\bar{\text{œ}}_{r,k}^i$ is the instantaneous change in desired differential mean orbit elements. The usual constraints on the weightings such that $0 \leq w_k^i, i = 1, \ldots, N_f$ and $w_k^1 + \cdots + w_k^{N_f} = 1$ are enforced. To be consistent with these constraints on individual spacecraft weights, a constraint on Δw^i must be enforced as $\sum_{i\in\mathcal{F}} \Delta w^i = 0$. For convenience, a vector containing individual spacecraft weights is introduced as

$$\mathbf{w}_k = \left[\ \cdots\quad w_k^i\quad \cdots\ \right]^T \tag{17}$$

Each spacecraft is subject to Keplerian orbit dynamics with J_2 perturbations. The mean orbit element propagation can be modeled in discrete time by the following dynamics

$$\bar{\text{œ}}_{k+1}^i = \mathbf{f}(\bar{\text{œ}}_k^i, k) \tag{18}$$

where $\mathbf{f}(\cdot)$ represents Keplerian orbit dynamics under the effect of J_2 perturbations. Spacecraft are also subject to impulsive maneuvers described by Equation (10). The $\Delta\mathbf{v}$ maneuvers are determined by each spacecraft's individual control laws and is typically a function of the state, $\mathbf{x}$, making $\Delta\mathbf{v}^i = \Delta\mathbf{v}^i(\mathbf{x}_k, k)$. Generally, the control law for each spacecraft is unknown. However, it should be assumed for the purposes of this formulation that individual $\Delta\mathbf{v}$ maneuvers satisfy the sufficiency condition explained in Lemma 1. With these state and decision variables, the reference formation system dynamics are defined by

$$\mathbf{x}_{k+1} = \begin{bmatrix} [\mathbf{w}_k]_{N_f \times 1} \\ \begin{bmatrix} \vdots \\ \delta\bar{\text{œ}}^i_{r,k} \\ \vdots \end{bmatrix}_{6N_f \times 1} \\ \begin{bmatrix} \vdots \\ \mathbf{f}\left(\bar{\text{œ}}^i_k + \mathbf{B}(\text{œ}^i)\Delta\mathbf{v}^i_k(\mathbf{x}_k, k), k\right) \\ \vdots \end{bmatrix}_{6N_f \times 1} \end{bmatrix} + \begin{bmatrix} \mathbf{I}_{N_f \times N_f} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{6N_f \times 6N_f} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \vdots \\ \Delta\mathbf{w}^i_k \\ \vdots \end{bmatrix} \\ \begin{bmatrix} \vdots \\ \Delta\delta\bar{\text{œ}}^i_{r,k} \\ \vdots \end{bmatrix} \end{bmatrix} \tag{19}$$

With the individual weights, mean reference orbits, and mean orbital elements in the state, the optimal control problem has all the information needed to fully describe the formation. With the formation fully described, any control laws which individual spacecraft act upon can be predicted and taken into account. Equation (19) can more conveniently be described as a nonlinear system that is affine in control.

$$\mathbf{x}_{k+1} = \mathbf{G}(\mathbf{x}_k, k) + \mathbf{B}\mathbf{u_k} \tag{20}$$

It now becomes clear that the choice to use incremental change in weightings and mean reference orbital elements as the control input, rather than the values themselves, was made in order to separate the control input from the formation state dynamics, $\mathbf{G}(\mathbf{x}_k, k)$.

In equation (19), the $\Delta\bar{\text{œ}}_k(\mathbf{x}_k, k)$ vector describes the impulsive control maneuvers performed by each spacecraft as dictated by their individual control laws. It must be emphasized that for the purposes of this formulation, this term is not part of the control input to the system. The formation system is to be controlled by its individual spacecraft weights and reference orbits only, so that the controller is not commanding $\Delta\mathbf{v}$ maneuvers explicitly. It is still required that the control laws be part of the formation system in order for the optimal control problem to accurately predict the motion of each spacecraft.

With the system and dynamics defined, the full general optimal formation control problem can be stated.

Formation Initialization/Maintenance OCP

$$\underset{\mathbf{u} \in \mathcal{U}_{adm}}{\text{minimize}} \qquad P = \sum_{i=1}^{N_f} \sum_{k=k_0}^{k_f} \alpha_i \|\Delta\mathbf{v}_{i,k}\|, \qquad \alpha^i \in [0, 1]$$

$$
\begin{aligned}
\text{subject to} \qquad & \mathbf{x}_{k+1} = \mathbf{G}(\mathbf{x}_k, k) + \mathbf{B}\mathbf{u_k} \\
& 0 \leq w_k^i \leq 1, \qquad\qquad i = 1, \ \ldots \ , N_f \\
& w_k^1 + \cdots + w_k^{N_f} = 1 \\
& \mathbf{g}\left(\mathbf{x}(k_0), \mathbf{x}(k_f)\right) = \mathbf{0}
\end{aligned}
$$

$$
\begin{aligned}
\text{where} \quad \mathbf{g}\left(\mathbf{x}(k_0), \mathbf{x}(k_f)\right) &= \left[\mathbf{0}_{6N_f \times N_f} \ \mathbf{I}_{6N_f \times 6N_f}\right] \mathbf{x}(k_f) - \delta\tilde{\text{œ}}_{r,k_f} \\
\delta\tilde{\text{œ}}_{r,k_f} &= \left[\cdots \left[\delta\bar{\text{œ}}^i_{r,k_f}\right]^T \cdots \left[\delta\bar{\text{œ}}^j_{r,k_0}\right]^T \cdots\right]^T, \qquad i \in N_Q \ , \ j \in N_M \\
\mathcal{U}_{adm} &= \left\{\mathbf{u} \middle| \sum \Delta w^i = 0 \ , \ 0 < \tilde{w}_k^i + \Delta\tilde{w}_k^i < 1\right\}
\end{aligned}
$$

This problem is quite difficult to solve due to the nonlinearities in $\mathbf{G}(\mathbf{x}_k, k)$ and the dimensionality of the state and control variables. It is also unrealistic to solve this control problem in real time on board each spacecraft, meaning that were a solution to be found, calculations would most likely be done on the ground and uploaded to the spacecraft.

While the full optimal control problem is difficult to solve, it might be possible to solve a more restricted problem while still gaining insight about optimal distributed formation control. If the problem is restricted to solving an optimization at each time step, rather than considering all possible future states of the formation, it is vastly simplified to something more manageable. The control input $\delta\bar{\text{œ}}_r$ is then made trivial given the boundary conditions. Two such optimization problems are presented which seek to answer two different questions.

Minimum Lyapunov Error Formations

It may be useful to know which weighting scheme will minimize the Lyapunov error function in Eqn. (4) if you have no knowledge of each spacecraft's control policy (i.e. $\Delta\mathbf{v}^i(\mathbf{x}_k, k)$). This is useful because it minimizes the total formation error.

Lemma 2. ***Minimum Lyapunov Error Formation***

Given a formation $\mathcal{F}$ with N_f spacecraft, each with differential mean orbit element errors, $\delta\bar{\mathbf{e}}^i$, as defined in Eqn. (3)*. The weighting scheme which minimizes the formation Lyapunov error function given in Eqn.* (4) *is always a Democratic (i.e. $w^i = 1/N_f \ \forall i \in \mathcal{F}$) formation, and is independent of time, state, or control policy.*
Proof. *The expression for differential mean orbit element error can be rewritten as*

$$
\begin{aligned}
\delta\bar{\mathbf{e}}^i &= \delta\bar{\text{œ}}^i_{r,k} - \delta\bar{\text{œ}}^i \\
&= \delta\bar{\text{œ}}^i_{r,k} - (\bar{\text{œ}}^i - \bar{\text{œ}}^b) \\
&= \delta\bar{\text{œ}}^i_{r,k} - \bar{\text{œ}}^i + \sum_{l \in \mathcal{F}} w^l(\bar{\text{œ}}^l - \delta\bar{\text{œ}}^l_{r,k}) \\
&= -\bar{\text{œ}}^{b,i} + \sum_{l \in \mathcal{F}} w^l \bar{\text{œ}}^{b,l}
\end{aligned} \tag{21}
$$

Substituting Equation (21) *into Equation* (4) *produces*

$$
\begin{aligned}
V &= \frac{1}{2}\sum_{i\in\mathcal{F}}\left[\delta\bar{\mathbf{e}}^{i}\cdot\delta\bar{\mathbf{e}}^{i}\right] \\
&= \frac{1}{2}\sum_{i\in\mathcal{F}}\left[\left(-\bar{\boldsymbol{\oe}}^{b,i}+\sum_{j\in\mathcal{F}}w^{j}\bar{\boldsymbol{\oe}}^{b,j}\right)\cdot\left(-\bar{\boldsymbol{\oe}}^{b,i}+\sum_{j\in\mathcal{F}}w^{j}\bar{\boldsymbol{\oe}}^{b,j}\right)\right]
\end{aligned}
\tag{22}
$$

In order to simplify Equation (22)*, the summation can be rewritten as a product of two matrices by using Equation* (17) *and defining*

$$
\boldsymbol{\Psi} = \begin{bmatrix} \bar{\boldsymbol{\oe}}^{b,1} & \cdots & \bar{\boldsymbol{\oe}}^{b,N_f} \end{bmatrix}_{6\times N_f} \tag{23}
$$

$$
\mathbb{1} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^{T} \tag{24}
$$

which allows certain summations to be rewritten as

$$
\sum_{i\in\mathcal{F}} w^{i}\bar{\boldsymbol{\oe}}^{b,i} = \bar{\boldsymbol{\oe}}^{b} = \Psi\mathbf{w} \qquad \sum_{i\in\mathcal{F}}\bar{\boldsymbol{\oe}}^{b,i} = \Psi\mathbb{1} \tag{25}
$$

Carrying out the substitution and expanding the dot product brings the expression of V to one which can be easily differentiated by $\mathbf{w}$

$$
\begin{aligned}
V &= \frac{1}{2}\sum_{i\in\mathcal{F}}\left[\left(-\bar{\boldsymbol{\oe}}^{b,i}+\boldsymbol{\Psi}\mathbf{w}\right)\cdot\left(-\bar{\boldsymbol{\oe}}^{b,i}+\boldsymbol{\Psi}\mathbf{w}\right)\right] \\
&= \frac{1}{2}\sum_{i\in\mathcal{F}}\left[\bar{\boldsymbol{\oe}}^{b,i}\cdot\bar{\boldsymbol{\oe}}^{b,i}-2\bar{\boldsymbol{\oe}}^{b,i}\cdot\boldsymbol{\Psi}\mathbf{w}+\boldsymbol{\Psi}\mathbf{w}\cdot\boldsymbol{\Psi}\mathbf{w}\right] \\
&= \frac{1}{2}\sum_{i\in\mathcal{F}}\left[\bar{\boldsymbol{\oe}}^{b,i}\cdot\bar{\boldsymbol{\oe}}^{b,i}\right]-\frac{1}{2}\sum_{i\in\mathcal{F}}\left[2\bar{\boldsymbol{\oe}}^{b,i}\right]\cdot\boldsymbol{\Psi}\mathbf{w}+\frac{1}{2}\sum_{i\in\mathcal{F}}\left[\boldsymbol{\Psi}\mathbf{w}\cdot\boldsymbol{\Psi}\mathbf{w}\right] \\
V &= \frac{1}{2}\sum_{i\in\mathcal{F}}\|\bar{\boldsymbol{\oe}}^{b,i}\|^{2}-\boldsymbol{\Psi}\mathbf{w}\cdot\boldsymbol{\Psi}\mathbb{1}+\frac{1}{2}N_f\boldsymbol{\Psi}\mathbf{w}\cdot\boldsymbol{\Psi}\mathbf{w}
\end{aligned}
\tag{26}
$$

Now that V is in a form which is written explicitly in terms of $\mathbf{w}$, Equation (26) *can be differentiated by $\mathbf{w}$ in order to solve for the weighting scheme which minimizes it.*

$$
\begin{aligned}
\frac{\partial V}{\partial\mathbf{w}} = \mathbf{0} &= -\frac{\partial\boldsymbol{\Psi}\mathbf{w}}{\partial\mathbf{w}}\boldsymbol{\Psi}\mathbb{1}+N_f\frac{\partial\boldsymbol{\Psi}\mathbf{w}}{\partial\mathbf{w}}\boldsymbol{\Psi}\mathbf{w} \\
&= -\boldsymbol{\Psi}^{T}\boldsymbol{\Psi}\mathbb{1}+N_f\boldsymbol{\Psi}^{T}\boldsymbol{\Psi}\mathbf{w}
\end{aligned}
\tag{27}
$$

Solving for $\mathbf{w}$ gives

$$
\begin{aligned}
N_f\boldsymbol{\Psi}^{T}\boldsymbol{\Psi}\mathbf{w} &= \boldsymbol{\Psi}^{T}\boldsymbol{\Psi}\mathbb{1} \\
\mathbf{w}^{*} &= \frac{1}{N_f}\mathbb{1}
\end{aligned}
\tag{28}
$$

which shows that the Democratic formation is the sole critical point of the Lyapunov function. To prove that it is indeed a minimizer, the second order necessary condition is used,

$$
\begin{aligned}
\frac{\partial^{2}V}{\partial\mathbf{w}^{2}} &= \mathbf{0}+N_f\frac{\partial\boldsymbol{\Psi}^{T}\boldsymbol{\Psi}\mathbf{w}}{\partial\mathbf{w}} \\
&= N_f\boldsymbol{\Psi}^{T}\boldsymbol{\Psi}
\end{aligned}
\tag{29}
$$

In order to show that Equation (29) *is positive semi-definite, it can be rewritten as*

$$\begin{aligned}
\frac{\partial^2 V}{\partial \mathbf{w}^2} &= N_f \mathbf{\Psi}^T \mathbf{\Psi} \\
&= N_f \begin{bmatrix} \bar{\text{œ}}^{b,1,T} \\ \vdots \\ \bar{\text{œ}}^{b,N_f,T} \end{bmatrix} \begin{bmatrix} \bar{\text{œ}}^{b,1} & \cdots & \bar{\text{œ}}^{b,N_f} \end{bmatrix} \\
&= N_f \begin{bmatrix} \bar{\text{œ}}^{b,1} \cdot \bar{\text{œ}}^{b,1} & \cdots & \bar{\text{œ}}^{b,1} \cdot \bar{\text{œ}}^{b,N_f} \\ \vdots & \ddots & \vdots \\ \bar{\text{œ}}^{b,N_f} \cdot \bar{\text{œ}}^{b,1} & \cdots & \bar{\text{œ}}^{b,N_f} \cdot \bar{\text{œ}}^{b,N_f} \end{bmatrix}
\end{aligned} \tag{30}$$

and since $N_f > 0$ *and the resulting matrix is positive semi-definite, the Democratic formation is the global minimizer of the Lyapunov function.* □

Using a Democratic formation is, in general, good, because Δv cost is commonly monotonic with $\delta\bar{\mathbf{e}}$. Therefore, subject to orbit element scaling, Democratic formations will tend to have lower Δv costs. It should be emphasized that no assumptions were made about the mean orbit elements or errors and, therefore, this result holds for all errors and mean orbit elements. However, it does not take into account the control policies, transformation matrix, $\mathbf{B}(\text{œ})$, or fuel cost. The following formulation attempts to account for these.

Minimum $\Delta\mathbf{v}$ cost

If the control policies of each individual spacecraft are known, it is possible to formulate a problem which can be solved to reduce the amount of $\Delta\mathbf{v}$ required to minimize formation error. This problem is independent of individual control policies, as long as they are known. The control policies can also be heterogeneous, meaning that individual spacecraft can have different control policies. $\Delta\mathbf{v}$ for each time step is a function of the weights of all spacecraft as well as their errors. If the cost function that is to be minimized is equal to

$$\Delta v_{tot,k} = P_k = \sum_{i \in \mathcal{F}} \|\Delta\mathbf{v}_k^i(\mathbf{x}_k, k)\| \tag{31}$$

which can be rewritten as

$$\Delta v_{tot,k} = P_k = \sum_{i \in \mathcal{F}} \|\Delta\mathbf{v}_k^i(w_k^i, \delta\mathbf{e}_k^i)\| \tag{32}$$

then the problem becomes: Given $\bar{\text{œ}}$ of all spacecraft in the formation, what is the weighting scheme which minimizes fuel usage. This gives a feedback controller for the weights which locally minimizes the total Δv at time k. Including the constraints on w^i, the cost function becomes

$$P_k = \sum_{i \in \mathcal{F}} \|\Delta\mathbf{v}_k^i(w_k^i, \delta\mathbf{e}_k^i)\| + \boldsymbol{\lambda}_k^T \boldsymbol{w}_k + \mu_k \left(\sum_{i \in \mathcal{F}} w_k^i - 1 \right) \tag{33}$$

where the second term represents the $w^i < 1\ \forall i \in \mathcal{F}$ constraint, with $\boldsymbol{\lambda}^T$ being a vector of Lagrange multipliers, and the third term represents the $w^i + \cdots + w^{N_f} = 1$ constraint, where μ is its corresponding Lagrange multiplier. For this particular formulation, it is desired that the formation error be maximally reduced. Therefore, the minimum Lyapunov error $\Delta\mathbf{v}^*$ from Equation (14) is used. The feedback controller is shown in the following equation:

$$\mathbf{w}_k^* = \arg\min_{\mathbf{w}} \left(P_k = \sum_{i\in\mathcal{F}} \|\mathbf{B}(\text{œ}^i)^+ \Delta\bar{\text{œ}}_k^{i,*}(w_k^i, \delta\bar{\mathbf{e}}_k^i)\| + \boldsymbol{\lambda}_k^T \boldsymbol{w}_k + \mu_k \left(\sum_{i\in\mathcal{F}} w_k^i - 1 \right) \right) \tag{34}$$

Note, for minimum Lyapunov error

$$\Delta\mathbf{v}_k^{i,*}(\mathbf{x}_k, k) = -\frac{1}{k(w^i, N_f)} \mathbf{B}(\text{œ}^i)^+ \left(\delta\bar{\mathbf{e}}_k^i - w_k^i \sum_{j\in\mathcal{F}} \delta\bar{\mathbf{e}}_k^j \right) \tag{35}$$

This parametric optimization routine can be solved numerically for its minimum value with respect to $\mathbf{w}$. The result will be a weighting scheme which minimizes total Δv cost. Although this method is calculated as if all spacecraft were moving simultaneously, it should also be effective at reducing the Δv cost for a formation moving asynchronously as well. The $\Delta\mathbf{v}$ maneuver performed with the resulting weighting scheme should also maximally reduce the formation Lyapunov error.

A controller proposed by Schaub and Alfriend[11] involves performing spacecraft maneuvers to correct errors in oscillating orbit elements, δa, δe, $\delta\omega$, and δM, at both the apoapses and periapses of the orbits. These maneuvers are defined by

Radial direction :

$$\Delta v_{rp} = -\frac{na}{4}\left\{ \frac{(1+e)^2}{\eta}(\delta\omega + \delta\Omega \cos i) + \delta M \right\} \tag{36}$$

$$\Delta v_{ra} = -\frac{na}{4}\left\{ \frac{(1-e)^2}{\eta}(\delta\omega + \delta\Omega \cos i) + \delta M \right\} \tag{37}$$

Along-track direction :

$$\Delta v_{sp} = \frac{na\eta}{4}\left[\frac{\delta a}{a} + \frac{\delta e}{(1+e)} \right] \tag{38}$$

$$\Delta v_{sa} = \frac{na\eta}{4}\left[\frac{\delta a}{a} - \frac{\delta e}{(1-e)} \right] \tag{39}$$

where the subscripts a and p denote maneuvers at the apoapsis and periapsis, respectively. This controller also performs maneuvers to reduce $\delta\Omega$ and δi errors at critical θ points on the orbits, which are defined as

$$\theta_c = \arctan\left(\frac{\delta\Omega \sin i}{\delta i} \right) \tag{40}$$

with the maneuvers defined as

$$\Delta v_w = \frac{h}{r}\sqrt{\delta i^2 + \delta\Omega^2 \sin i^2} \tag{41}$$

This controller is henceforth referred to as $\Delta\mathbf{v}^{i,SA}(\mathbf{x}_k, k)$.

RESULTS

The 'SA' impulsive feedback control approach is used as a baseline for the formation control comparison. However, this control approach is designed to control one spacecraft's orbit and takes no consideration of an entire formation's error. This would be acceptable for a special case of

Case	Title	w^1	w^2	w^3	Description
TC1.	L/F SA (Schaub-Alfriend)	1	0	0	Using Schaub's impulsive control approach
TC2.	L/F SA†	1	0	0	
TC3.	L/F SA†	0	1	0	
TC4.	L/F SA†	0	0	1	
TC5.	D SA	1/3	1/3	1/3	
TC6.	D SA†	1/3	1/3	1/3	
TC7.	D $\Delta\mathbf{v}^*$†	1/3	1/3	1/3	Impulse by $\Delta\mathbf{v}^*$ at apsides and nodes
TC8.	L/F $\Delta\mathbf{v}^*$†	1	0	0	
TC9.	L/F $\Delta\mathbf{v}^*$†	0	1	0	
TC10.	L/F $\Delta\mathbf{v}^*$†	0	0	1	
TC11.	Feedback Controller $\Delta\mathbf{v}^*$†	w_k^1	w_k^2	w_k^3	Feedback controller active

Table 1. Test cases run. †denotes the sufficient condition for formation stable maneuvers is enforced

a Leader/Follower formation, but is not ideal for a formation with varying weights. Because of this, two simulation cases are made to compare what happens when Schaub's impulsive controls approach is used on a L/F formation and a Democratic formation.

Further building on concepts presented in this paper, the sufficiency condition for stable uncoordinated maneuvers outline in Lemma 1 is enforced on a Democratic formation, using Schaub's impulsive control approach, in a third test case.

A new control approach attempting to capture the minimum Lyapunov error $\Delta\bar{\text{œ}}^{i,*}$ is developed. With this control approach, impulsive control maneuvers calculated using Eqn. (35) are performed at each spacecraft's apoapsis, periapsis, and nodes. With this new control policy, spacecraft maneuver in order to reduce formation error as a whole, rather than their individual error. Therefore, the formation should converge to steady-state faster.

Finally, the feedback controller introduced in Eqn. (34) is implemented onto a set of spacecraft which each use the same control policy as TC7. In order to investigate the optimality of this feedback controller, three test cases are simulated which pick each of the three spacecraft as the leader in L/F formations. All of the test cases are summarized in Table 1.

The simulation was developed using Matlab's `ode45` function. `ode45` was used to propagate each spacecraft forward 10 seconds using two-body dynamics in the ECI frame with J_2 perturbations, with $k_f = 24hrs$. Each spacecraft then computes its estimate of the weighted formation barycenter using Eq. (1). Finally each spacecraft computes their orbit error using Eq. (3). The formation slots that are used in each simulation are given in Table 1(a). And the initial conditions for each spacecraft is shown in Table 1(b).

Figure 1 shows the output for TC1. It shows that the 'SA' impulsive maneuvering approach works well for a L/F formation. Most of the major maneuvers are completed within 4 orbits. Note that because SC1 is the leader, it has no error and therefore does not maneuver. Figure 2 shows TC1 again, only this time in a log-space format in order to better show the errors as they become small. Log-space plots will be used for the remainder of the simulation Test Cases. Figure 4(a) shows the impulsive control approach working on a democratic formation. This output shows that

(a) Desired formation slots for each simulation

$\delta\bar{œ}^i_{r,k}$	SC1	SC2	SC3
$\delta\bar{a}(km)$	$3.0e^{-3}$	$3.9e^{-3}$	$-1.5e^{-3}$
$\delta\bar{e}$	$-2.7e^{-3}$	$-3.4e^{-3}$	$1.3e^{-3}$
$\delta\bar{i}(rad)$	$2.9e^{-4}$	$-4.6e^{-4}$	$1.8e^{-4}$
$\delta\bar{\omega}(rad)$	$4.6e^{-3}$	$-3.5e^{-3}$	$2.6e^{-3}$
$\delta\bar{\Omega}(rad)$	$1.6e^{-3}$	$-4.3e^{-3}$	$2.4e^{-3}$
$\delta\bar{M}(rad)$	0	0	0

(b) Initial conditions for each simulation

$\bar{œ}^i_0$	SC1	SC2	SC3
$\bar{a}_0(km)$	7500.9	7499.8	7500
$\bar{e}_0$	$7.4e^{-3}$	$6.59e^{-3}$	$1.12e^{-2}$
$\bar{i}_0(rad)$	0.350	0.348	0.349
$\bar{\omega}_0(rad)$	3.00	3.00	3.00
$\bar{\Omega}_0(rad)$	1.98	1.98	2.00
$\bar{M}_0(rad)$	0.220	0.221	0.200

Table 2. Desired formation slots and initial conditions for all Test Cases

the formation takes more time to decrease its Lyapunov error, but it ends up consuming less $\Delta\mathbf{v}$.

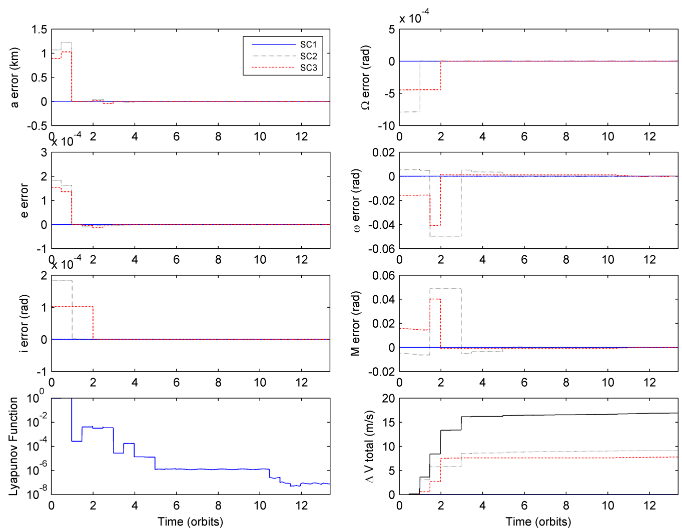


Figure 1. TC1: $w^1 = 1$, Schaub's impulse approach, without sufficiency condition

Figures 3(a) through 3(c) show Test Cases 2 through 4, which show each spacecraft being the leader under the SA control law, with the sufficient condition in Lemma 1 active. The effects of the sufficiency condition can be seen when these figures are compared with Figure 2.

Figure 4(b) shows the difference in orbit maneuvering with the sufficiency condition applied. It can be seen that with the sufficiency condition applied, the Lyapunov function never increases, and the formation uses less fuel than without it. However, the Lyapunov error is not reduced as much as in previous test cases. This Test Case was simulated longer to make sure that it had indeed 'settled' at the end of the 24-hour run, but the 24-hour duration was kept for comparison purposes. It must be stressed that the 'SA' control approach is not intended to be constrained by Lemma 1, and that TC6 is only intended to show that this constraint is effective with any controller.

Figure 4 shows TC7 error results. This not only achieves the greatest error reduction using the smallest $\Delta\mathbf{v}$, but also does it in a comparatively short time. Because the pseudo-inverse is an

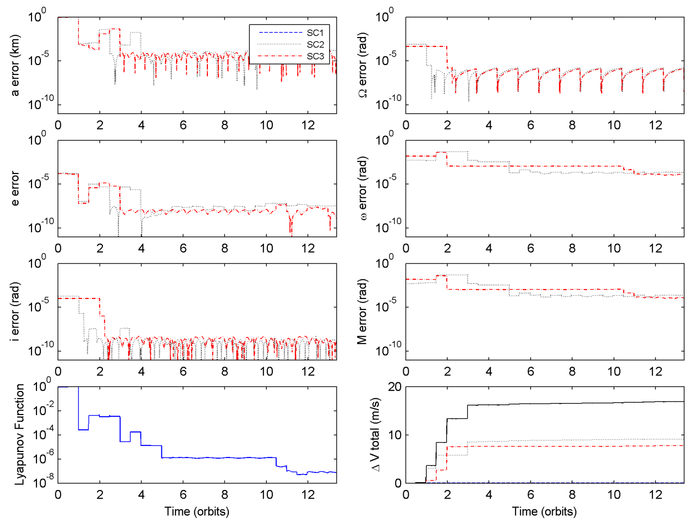


Figure 2. TC1: $w^1 = 1$**, Schaub's impulse approach, without sufficiency condition**

approximation, however, the error will never be fully reduced and tends to remain mostly constant on the order of 10^{-5} for each orbital element.

Feedback Formation Control

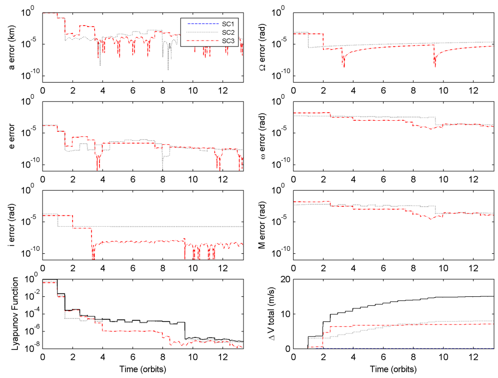
This simulation uses Eqn. (35) for all three spacecraft control laws. Before each maneuver is performed, Matlab's `fmincon` function is used to determine the weighting scheme which minimizes Equation (32). $\bar{\text{œ}}^b$, $\delta\bar{e}$, and $\Delta\bar{\text{œ}}^{i,*}$ are then recalculated based on the new weights, and the maneuver is performed.

Figures 5(a), 5(b), and 5(c) show the simulations for Test Cases 8, 9, and 10, respectively. These figures show that arbitrarily picking a the leader in a formation can lead to different Δv costs, while leading to similar Lyapunov errors.

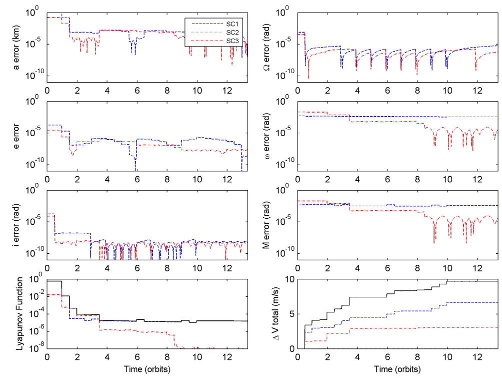
When these three figures are compared with Figure 4, it can be surmised that, with no prior knowledge of the future formation states, a Democratic formation is a safe choice. This further reinforces the applicability of Lemma 2. This is an important result, because it means that with no *a-priori* knowledge of the formation state or control laws, picking the formation to be Democratic will more than likely not be the worst-case scenario.

Figure 6 shows the error history of the mean orbit element errors using the minimum Δv cost feedback formation controller. It can be seen that $\Delta\mathbf{v}$ cost has been reduced when compared to TC7-TC10. However, the minimum Δv cost feedback controller is not optimal, and it is not expected that it will always produce better results than Democratic or L/F formations. It should be noted that the sufficiency condition in Lemma 1 does not account for changes in weights, and therefore the Lyapunov function may increase as the weights are altered. The formation error Lyapunov function and Δv_{tot} at time k_f for all Test Cases are summarized in Table 3.

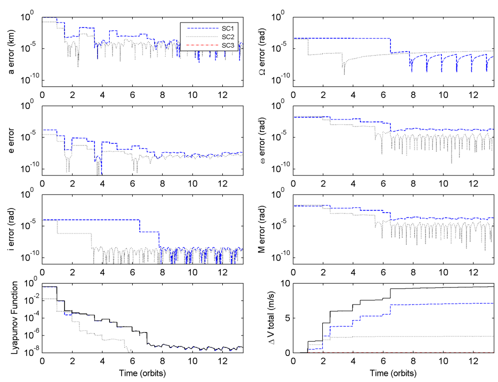
Figure 7 shows the feedback formation control $\mathbf{w}_k$ of the formation vs. time. The chosen weighting scheme switches between L/F formations with different spacecraft being chosen as the leader.



(a) TC2: $w^1 = 1$

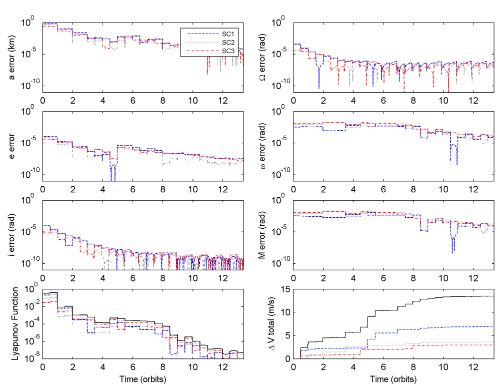


(b) TC3: $w^2 = 1$

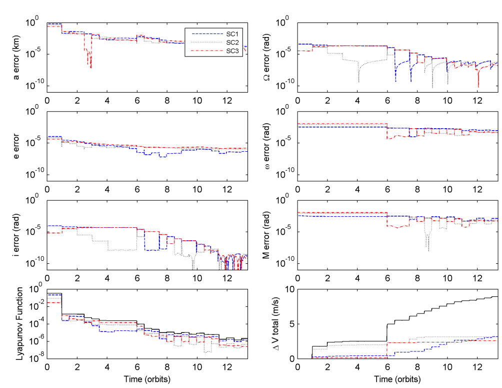


(c) TC4: $w^3 = 1$

Figure 3. Leader/Follower formations with each spacecraft using SA with active sufficiency condition



(a) TC5: Democratic weights, Schaub's impulse approach, without sufficiency condition



(b) TC6: Democratic weights, Schaub's impulse approach, with sufficiency condition

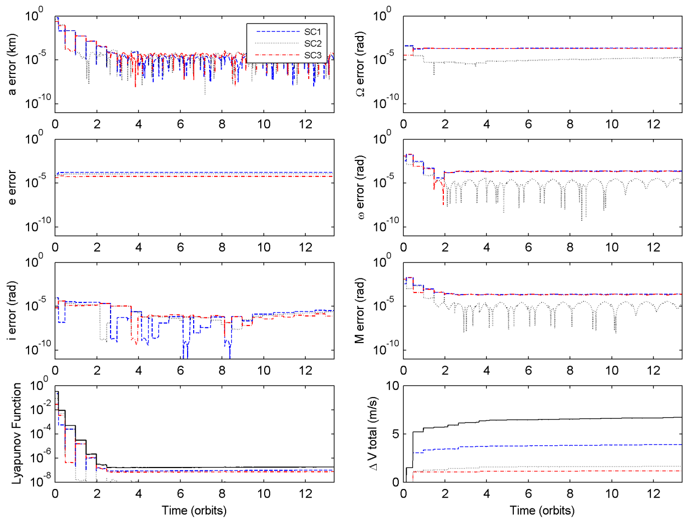
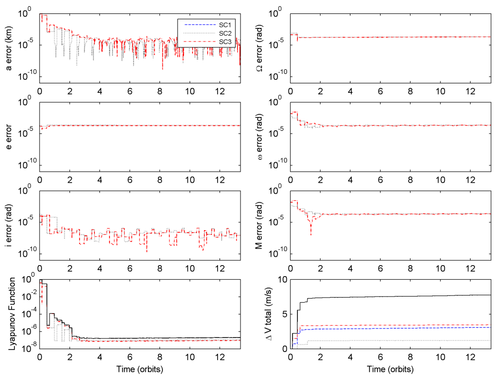
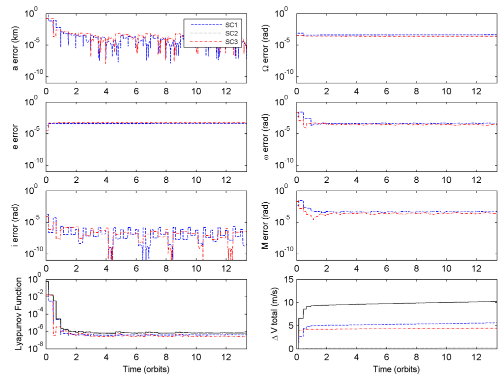


Figure 4. TC7: Democratic weights, Δv^*, with sufficiency condition

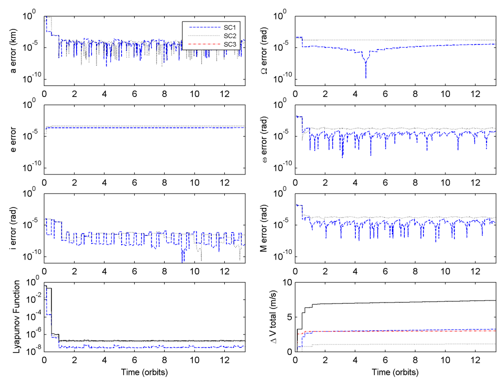
Figures 8(a) and 8(b) shows a Ternary plot of the weighting scheme field early in the simulation, with the value of Δv_{tot} as the z axis. From this figure the topology of the weighting scheme can be studied. It can be seen that there is a clear best choice for the leader, as well as a worst choice, and the third leadership position having a Δv cost somewhere in between. Figures 8(c) and 8(d) shows a Ternary plot at a time later in the simulation when "steady-state" has been achieved. The topology is similar to the early simulation Ternary plot, with one spacecraft being the clear choice for leader. The values of P are lower than in Figure 8(a), as expected since the total formation error has been reduced by this point. It can be seen that in both plots, the Democratic formation, which is in the center of the plot, is a saddle point. Again, this appears to be a consequence of Lemma 2.



(a) TC8: $w^1 = 1$



(b) TC9: $w^2 = 1$



(c) TC10: $w^3 = 1$

Figure 5. Leader/Follower formations with each spacecraft using $\Delta \mathbf{v}^*$ with active sufficiency condition

Case	Title	V	$\Delta v_{tot}(m/s)$
TC1.	L/F SA (Schaub-Alfriend)	$7.52e^{-8}$	16.90
TC2.	L/F SA†	$6.81e^{-8}$	15.04
TC3.	L/F SA†	$1.53e^{-5}$	9.6947
TC4.	L/F SA†	$3.92e^{-8}$	9.5029
TC5.	D SA	$5.49e^{-8}$	13.51
TC6.	D SA†	$2.12e^{-6}$	8.92
TC7.	D $\Delta \mathbf{v}^*$†	$1.81e^{-7}$	6.70
TC8.	L/F $\Delta \mathbf{v}^*$†	$2.01e^{-7}$	7.75
TC9.	L/F $\Delta \mathbf{v}^*$†	$6.65e^{-7}$	10.21
TC10.	L/F $\Delta \mathbf{v}^*$†	$1.91e^{-7}$	7.38
TC11.	Feedback Controller $\Delta \mathbf{v}^*$†	$2.43e^{-7}$	6.99

Table 3. Summary of results at time k_f

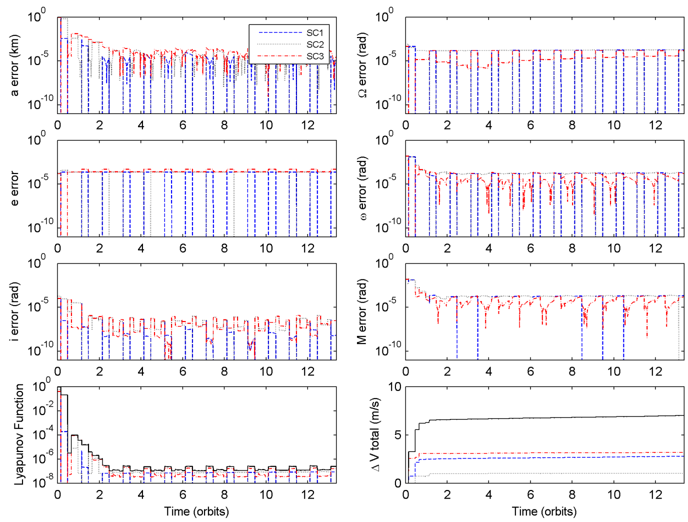


Figure 6. TC11: $\triangle v^*$ with parametric optimization active

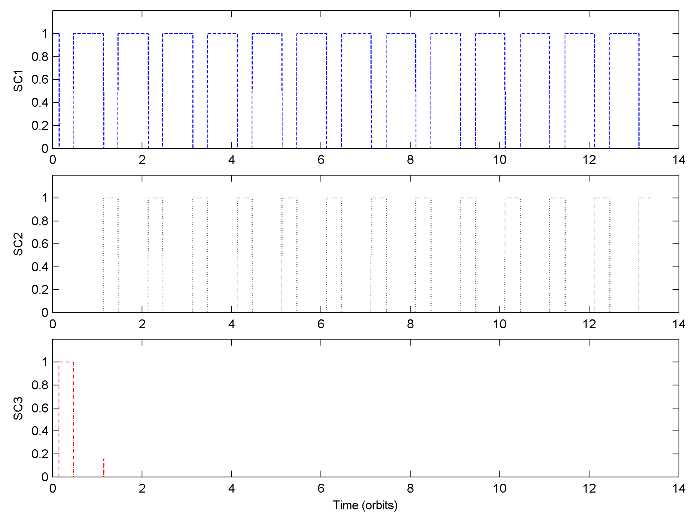
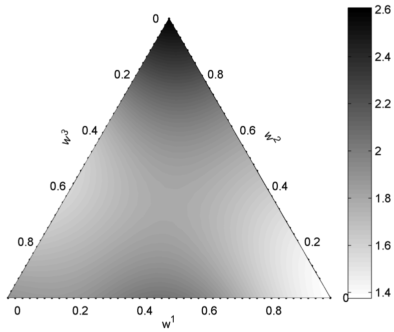
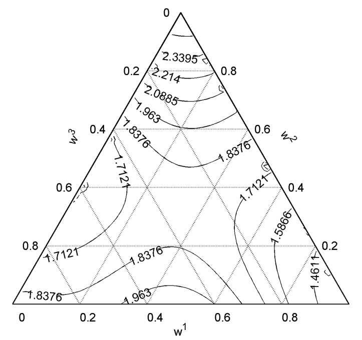


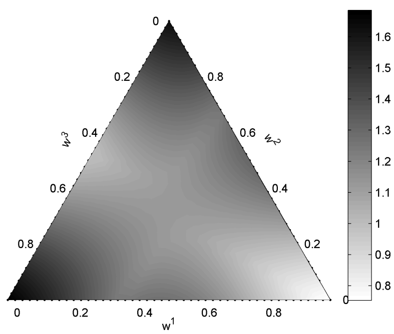
Figure 7. TC11: Weighting scheme vs. time



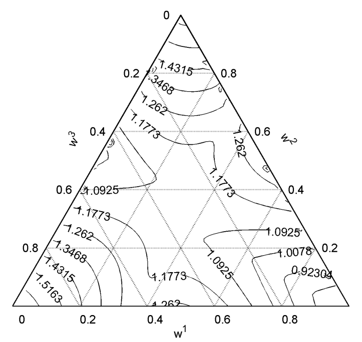
(a) Continuous. $t_k = 3020s$, apoapsis maneuver



(b) Contour. $t_k = 3020s$, apoapsis maneuver



(c) Continuous. $t_k = 77290s$, periapsis maneuver



(d) Contour. $t_k = 77290s$, periapsis maneuver

Figure 8. TC11: Ternary map of Δv_{tot} values with respect to weighting schemes at two different points in the simulation

CONCLUSIONS

In this paper, a framework for individual spacecraft control as outlined in previous works was used to develop a general optimal formation control problem. Due to the difficulty involved in solving this problem, including dimensionality and non-linearity, more restricted problems were proposed to answer questions about the what the best weighting scheme is under certain conditions. The first of which asks what weighting scheme will minimize the orbit error Lyapunov function given. It is found that this is essentially a type of balancing problem where, without making assumptions on the errors or control policies of the spacecraft, the optimal solution is to distribute the weight evenly between each spacecraft. Next, a method of calculating the minimum Δv cost at each time step given a set of spacecraft with arbitrary control policies is formulated. However, this solution is not optimal because it does not take into account the entire forward time history of the formation state. It is found that a Democratic formation is a 'safe' choice if resources to solve the Optimal Formation Control Problem or the minimum Δv feedback formation controller. Also, using the feedback controller performs approximately as well as the best constant-leader L/F case, and it performs better than the worst constant-leader L/F case. Therefore, using the feedback controller is preferred to simply picking a L/F formation. Simulations demonstrating these concepts were made. Future works may include developing weighting schemes which take further orbit propagation into account, creating a more optimal weighting solution, as well as investigations on fuel balancing between the spacecraft in formation.

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